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Group Contraction and the Nine Cayley–Klein Geometries[1,2]

M. A. Fernández Sanjuan

Departamento de Física Teórica, Facultad de Ciencias, Universidad de Valladolid, Valladolid, Spain

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A one-to-one correspondence between the nine Cayley–Klein geometries and the so-called kinematic groups is presented. As is well known, the kinematic groups are related to each other through group contraction. The pattern of contraction is explained by relating each kinematic group to a specific Cayley–Klein geometry. The very meaning of group contraction is deeply rooted in the relationship between the nine geometries. Lie algebras of those geometries are explicitly constructed.

1. INTRODUCTION

Group theory is a mathematical tool in physics that not only expresses invariance, but also acts as "superlaws" determining the possible form of physical laws yet unknown (Bacry and Levy-Leblond, 1968). Starting from very general physical principles, one is able to classify all possible kinematics, compatible with a relativity principle dictated by very general assumptions.

On the other hand, looking at the scheme of contractions of the kinematic groups, one can realize that there exist groups—some so-called relative-time groups, i.e., the de Sitter and Poincaré and other absolute-time groups such as Newton–Hooke and Galilei—that can be related to each other through the process of "group contraction" which gives a new meaning to each group.

Thus it is well known that the Galilei group is a contraction of the Poincaré group, but also is a contraction of the Newton–Hooke groups, providing each group with a very rich structure.

[1] In part from the Tesina de Licenciatura

[2] M. A. Fernández Sanjuan, Tesina de Licenciatura: "Contracciones anisótropas del grupo de Poincaré," Universidad de Valladolid, 1981.

One can ask the reason for the relationship existing between the kinematic groups.

In this paper we study the Lie algebras of the nine plane Cayley–Klein geometries (Yaglom, 1979) and their relations to each other through group contraction. This gives the real foundations of this subject.

We reinterpret the relationship among the kinematic groups, looking at it as a geometrical relation between the underlying plane Cayley–Klein geometries.

The organization of this paper is as follows: In Section 2 a brief summary of the nine Cayley–Klein geometries is presented. Also a construction of the Lie algebras and the corresponding group contractions of geometries are given. In Section 3 the unidimensional kinematic groups and the related contractions are studied; and finally in Section 4 the principal conclusions of this work are stated.

2. THE NINE CAYLEY–KLEIN GEOMETRIES

In accordance with the Erlangen Program (Klein, 1974), due to Felix Klein, each geometry is associated with a group of transformations, and hence there are as many geometries as groups of transformations.

Associated with the group of transformations that in physics guarantees the invariance of many mechanical systems, the Galilei group, is the so-called Galilean geometry that, as has been pointed out by Yaglom, "in spite of its relative simplicity, confronts the uninitiated reader with many surprising results."

We define all the plane geometries that were first introduced by Klein in 1871.

Following Cayley and Klein, we distinguish three fundamentally different geometries on a line, depending on the way of measuring length: Euclidean geometry, elliptic geometry, and hyperbolic geometry.

Just as there are three ways of measuring length that give rise to three geometries on a line, so there are three ways of measuring angles that gives

Table 1. The Nine Cayley–Klein Geometries.

Measure of angles	Measure of lengths		
	Elliptic	Parabolic	Hyperbolic
Elliptic	Elliptic	Euclidean	Hyperbolic
Parabolic	co-Euclidean	Galilean	co-Minkowski
Hyperbolic	co-hyperbolic	Minkowski	Doubly hyperbolic

rise to three geometries in a pencil of lines. Then, the Cayley–Klein scheme yields nine plane geometries.

There is a simple method to obtain from each geometry a new one. All we need to do is to call each point a line and each line a point, and, of course, call the distance between points the angle between lines, and call the angle between lines the distance between points; that is, to use the duality transformation. Thus, the elliptic, Galilean, and doubly hyperbolic geometry are duals to themselves. Euclidean, Minkowskian, and hyperbolic geometries possess their own dual geometries or co-geometries: co-Euclidean, co-Minkowski, and co-hyperbolic geometries.

An analytical study of the nine geometries can be made. As is well known, the points of the Euclidean plane can be identified with the complex numbers associated with each point; it is possible to generalize this study, which at first glance will be valid for elliptic, Euclidean, and hyperbolic geometries, by considering, in addition to complex numbers, others: the duals and double numbers.

Thus we can associate the complex numbers with points of the geometries with elliptic angular metric, the dual numbers with points of the geometries with parabolic angular metric, and the double numbers with the points with hyperbolic angular metric.

To discover the Lie algebra of each geometry it is useful to know the group of motions of each geometry.

The analytical expressions of the isometries are as follows: If the metric of distances is elliptic

$$z' = \frac{pz + q}{\bar{q}z - \bar{p}}, \qquad |p\bar{p} + q\bar{q}| \neq 0 \tag{1}$$

if the metric is parabolic

$$z' = pz + q, \qquad |p\bar{p}| \neq 0 \tag{2}$$

and finally if the metric is hyperbolic

$$z' = \frac{pz + q}{\bar{q}z + \bar{p}}, \qquad |p\bar{p} - q\bar{q}| \neq 0 \tag{3}$$

where p and q are complex, duals, or double numbers that verify the conditions (1), (2), and (3).

A detailed study of those geometries can be found in Yaglom (1979).

2.1. The Lie Algebras. Now, knowing the structure of the group of motions of each geometry, we find the associated Lie algebras, using the standard method.

A quite well-known process in physics, the contraction of groups, relates those nine geometries among them in a very clear way that we explain in the following section.

We show the results obtained for each geometry, but before grouping the results, we examine one of the cases, the hyperbolic geometry, to demonstrate a particular case. The method is similar for the rest of the geometries. For the hyperbolic geometry we know beforehand the distance between two given points z and z_1, the motions, and the straight lines:

$$\text{distance: } \tanh^2 \frac{d_{z,z_1}}{2} = \frac{(z - z_1)(\bar{z} - \bar{z}_1)}{(1 - z\bar{z}_1)(1 - \bar{z}\bar{z}_1)}$$

$$\text{motions: } z' = \frac{pz + q}{\bar{q}z + \bar{p}}, \qquad z \in \mathbb{C}, \qquad |p\bar{p} - q\bar{q}| = 0$$

$$\text{lines: } Az\bar{z} + Bz - \bar{B}\bar{z} + A = 0, \qquad \operatorname{Re} A = 0$$

We want to study the group of motions and find its Lie algebra.

First of all, what is the law of the group?

$$z \to \frac{pz + q}{\bar{q}z + \bar{p}} \to \frac{p'\left(\dfrac{pz + q}{\bar{q}z + \bar{p}}\right) + q'}{q'\left(\dfrac{pz + q}{\bar{q}z + \bar{p}}\right) + \bar{p}'} = \frac{(p'p + q'q)z + p'q + \bar{p}q'}{(\bar{q}'p + \bar{q}\bar{p}')z + \bar{p}\bar{p}' + \bar{q}'q}$$

The parameters p and q are underdetermined because we can always change the choice of parameters to λp and λq with λ real.

Then

$$p'' = p'p + q'\bar{q}$$

$$q'' = p'q + \bar{p}q'$$

is the law of the group.

Through the association

$$(p, q) \to \begin{pmatrix} p & q \\ q & p \end{pmatrix} \qquad \text{with } |p\bar{p} - q\bar{q}| = 1$$

the law of the group is translated to the product of matrices.

Now we select a few kinds of transformations which generate the groups. We call the transformations which do not move the point z_0, *rotations around* z_0. On the other hand, Γ being a straight line which goes through z_0, we call the transformations which leave Γ invariant *translations along the line* Γ.

We study the translations along the line Γ and the rotations. If, for instance, we choose $z_0 = 0$, then the rotations take the form

$$R_p = \begin{pmatrix} p & 0 \\ 0 & \bar{p} \end{pmatrix}, \qquad |p| = 1, \qquad p = \exp(i\alpha/2)$$

and the translations, we can easily prove, are of the form

$$T_q = \begin{pmatrix} 1 & q \\ \bar{q} & 1 \end{pmatrix}, \qquad |1 - q\bar{q}| = 1, \qquad q = r\exp(i\alpha/2)$$

The translations T_r and $T_{re^{i\alpha/2}}$ move along lines which form an angle α between them.

Notice that (T_q) is not a subgroup, while (R_p) generates a uniparametric subgroup of parameter α.

Furthermore every motion is a product of a T_r and a R_p. Indeed,

$$T_r R_p = \begin{pmatrix} 1 & r \\ \bar{r} & 1 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & \bar{p} \end{pmatrix} = \begin{pmatrix} p & r\bar{p} \\ \bar{r}p & \bar{p} \end{pmatrix}$$

and it is sufficient to choose $r\bar{p} = q$.

To find the Lie algebra, we choose three uniparametric subgroups: R_p and the suitable translations to two fixed directions:

$$T_q = \begin{pmatrix} 1 & re^{i\alpha/2} \\ re^{-i\alpha/2} & 1 \end{pmatrix}$$

If we choose for convenience $\alpha/2 = 0$ and $\alpha/2 = \pi/2$, then we obtain the translations in two orthogonal directions:

$$T_1 = \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 1 & ir \\ -ir & 1 \end{pmatrix}, \qquad r = q/p$$

Every motion, as one can prove, is a product of three of these. We also have to find the canonical parameters of these uniparametric subgroups (Levy-Leblond, 1979).

For the rotations $p = e^{i\alpha/2}$, where α is canonical. Then

$$R = \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}$$

For the translations, the composition law of the subgroup is the following:

$$\begin{pmatrix} 1 & r' \\ r' & 1 \end{pmatrix}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} = \begin{pmatrix} rr'+1 & r+r' \\ r'+r & rr'+1 \end{pmatrix} \alpha \begin{pmatrix} 1 & \dfrac{r+r'}{1+rr'} \\ \dfrac{r+r'}{1+rr'} & 1 \end{pmatrix}$$

where we can see that the parameter r is not canonical but the change $r = \tanh(d/2)$ makes d canonical in such a way that

$$T_1 = \begin{pmatrix} 1 & \tanh(d/2) \\ \tanh(d/2) & 1 \end{pmatrix} \quad \text{and}$$

$$T_2 = \begin{pmatrix} 1 & i\tanh(d/2) \\ -i\tanh(d/2) & 1 \end{pmatrix}$$

We write $d/2$ so that the distance traveled is the group parameter. Then the generators of these uniparametric subgroups are

$$K = \begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad \text{and}$$

$$P = \begin{pmatrix} 0 & i/2 \\ -i/2 & 0 \end{pmatrix}$$

where K is the infinitesimal generator of the rotations and H and P are the infinitesimal generators suitable to the translations.

When we know the generators, it is easy to get the Lie brackets of the Lie algebra of the group of motions, in this case of the hyperbolic geometry

$$[K, H] = P$$

$$[K, P] = -H$$

$$[H, P] = -K$$

We show the Lie algebras of the nine geometries in Table II.

We must make a point: according to the method we have developed to obtain the Lie algebra of each geometry, we can see that, essentially, there is one geometry, because we can call the generators of the algebra as we wish, without any previous idea about each of the symbols. Nevertheless, from a

Table II. Lie Algebras of the Geometries

Elliptic	Euclidean	Hyperbolic
$[K,H]=P$	$[K,H]=P$	$[K\,H]=P$
$[K,P]=-H$	$[K,P]=-H$	$[K\,P]=-H$
$[H,P]=K$	$[H,P]=0$	$[H\,P]=-K$
Co-Euclidean	Galilean	Co-Minkowski
$[K,H]=P$	$[K,H]=P$	$[K,H]=P$
$[K,P]=0$	$[K,P]=0$	$[K,P]=0$
$[H,P]=K$	$[H,P]=0$	$[H,P]=-K$
Co-Hyperbolic	Minkowski	Doubly hyperbolic
$[K,H]=P$	$[K,H]=P$	$[K,H]=P$
$[K,P]=H$	$[K,P]=H$	$[K,P]=H$
$[H,P]=K$	$[H,P]=0$	$[H,P]=-K$

physical point of view, there are different theories because we have a concrete interpretation of P and H associated with concrete physical transformations; P usually acts as the generator of the space translations while H acts as the generator of the time translations.

Thus, some of the previous geometries lead us to several cases, different from a physical point of view, but equivalent from a geometrical point of view.

Table III gives a complete diagram where one has to think of P, H, and K as generators of transformations which are not equivalent physically.

The data in Table III are those for the Lie brackets $[K, H]$, $[K, P]$, and $[H, P]$ in that order. In the trivial geometry all the Lie brackets vanish.

Table III. Lie Algebras for the Physically Amplified Geometries.

Elliptic			Euclidean		Hyperbolic	
P			P		P	
$-H$			$-H$		$-H$	
K			0		$-K$	
Co-Euclidean	co-Euclidean′	Galilean	Carroll	Translations	co-Minkowski	co-Minkowski′
P	0	P	0	0	P	0
0	$-H$	0	H	0	0	$-H$
K	K	0	0	$-K$	$-K$	$-K$
Co-hyperbolic			Minkowski		Doubly hyperbolic	
P			P		P	
H			H		H	
K			0		$-K$	

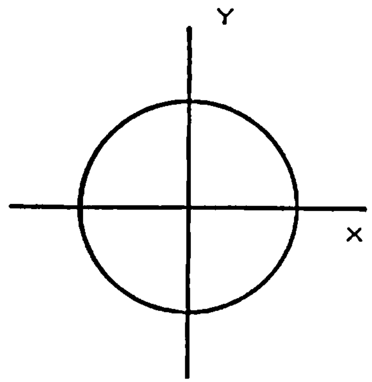

Fig. 1.

2.2. Contractions of the Geometries. Group contraction is a mathematical process first used by Inönu and Wigner (1953) to study the relationship between relativistic and classical mechanics (the classical limit of relativity).

From an intuitive point of view, we can see that Galilean geometry is an intermediate case of the Euclidean and Minkowskian geometries.

Let Figure 1 be a Euclidean circle. If we consider the unit of measure in the y axis much smaller than the unit of measure in the x axis and at the same time only consider small rotations, then that would be expressed graphically by drawing a small slice around the x axis (Figure 2). If we stretch it as though it were extensible, we have Figure 3 which is a Galilean circle.

We can carry out the same process with the "circles" of the Minkowskian geometry.

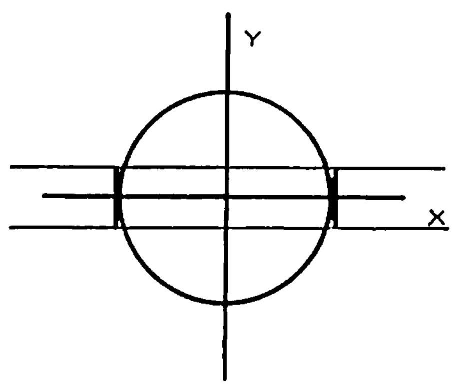

Fig. 2.

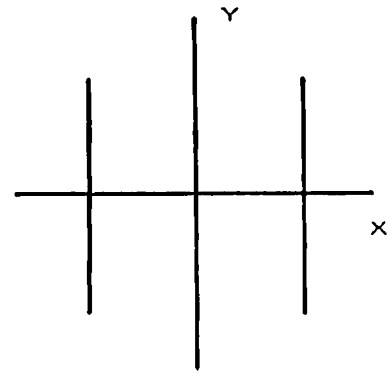

Fig. 3.

We are going to study the contractions of the geometries from a more analytical point of view, according to the theory of Lie algebras.

Essentially, we construct a one-parameter family of automorphisms of the algebra which is singular at a certain value of the parameter.

The general contraction will be

$$P \to \alpha P$$

$$H \to \beta H$$

$$K \to \gamma K$$

with α, β, and γ depending on a parameter ε so that $\alpha\beta\gamma \to 0$ at a value for the parameter we will take to be $\varepsilon = 0$.

The commutators of the Lie algebra must admit a limit for $\varepsilon \to 0$ and that limit must be finite.

Basically there are four types of contractions:

(i) Type PH:

$$P \to \varepsilon P, \qquad H \to \varepsilon H, \qquad K \to K$$

is called a spatial contraction and produces certain contracted geometries which have a local relationship to the ones we have contracted.

(ii) Type HK:

$$P \to P, \qquad H \to \varepsilon H, \qquad K \to \varepsilon K$$

is the contraction of direction 1 to an angle and through it we get the Carroll geometry from the geometry of Minkowski.

Table IV. Contractions of the Cayley–Klein Geometries

Contraction	El	Eu	H	Co-E	G	Co-M
	Elliptic	Euclidean	Hyperbolic	Co-Euclidean	Galilean	Co-Minkowski
	P	P	P	P	P	P
PH	$-H$	$-H$	$-H$	0	0	0
	0	0	0	0	0	0
gives	Eu	Eu	Eu	G	G	G
	0	0	0	0	0	0
KH	$-H$	$-H$	$-H$	0	0	0
	K	0	$-K$	K	0	$-K$
gives	co-E'	C	co-M'	T	St	T
	P	P	P	P	P	P
KP	0	0	0	0	0	0
	K	0	$-K$	K	0	$-K$
gives	co-E	G	co-M	co − E	G	co-M

(iii) Type PK:

$$P \to \varepsilon P, \qquad H \to H, \qquad K \to \varepsilon K$$

is the contraction of direction 2 to an angle and through it we can pass from the geometry of Minkowski to that of Galileo.

(iv) Type PHK:

$$P \to \varepsilon P, \qquad H \to \varepsilon H, \qquad K \to \varepsilon K$$

is the total contraction and it takes each geometry to the "static" one, the geometry where all Lie brackets vanish.

We show in Table IV the contractions that each geometry has suffered and the relationship between them. The interesting point here is that each geometry admits, besides its own interpretation, other approximate interpretations by virtue of the concept of group contraction.

When we know all the contractions of the different geometries, we can build a lattice of contractions where we can see the new approximate interpretation of each geometry (Figure 4).

3. KINEMATIC GROUPS

We summarize the results obtained by Bacry and Levy-Leblond concerning the possible structures of groups of transformations which relate two inertial systems under quite general physical hypotheses.

Table IV. *Continued*

Co-H	M	D-H	Co-E'	C	T	Co-M'
Co-Hyperbolic	Minkowski	Doubly Hyperbolic	Co-Euclidean′	Carroll	Translations	Co-Minkowski′
P	P	P	0	0	0	0
H	H	H	H	H	0	H
0	0	0	0	0	0	0
M	M	M	C	C	St	C
0	0	0	0	0	0	0
H	H	H	H	H	0	H
K	0	$-K$	$-K$	0	$-K$	K
co-M'	C	co-E'	co-E'	C	T	co-M'
P	P	P	0	0	0	0
0	0	0	0	0	0	0
K	0	$-K$	$-K$	0	$-K$	K
co-E	G	co-M	T	St	T	T

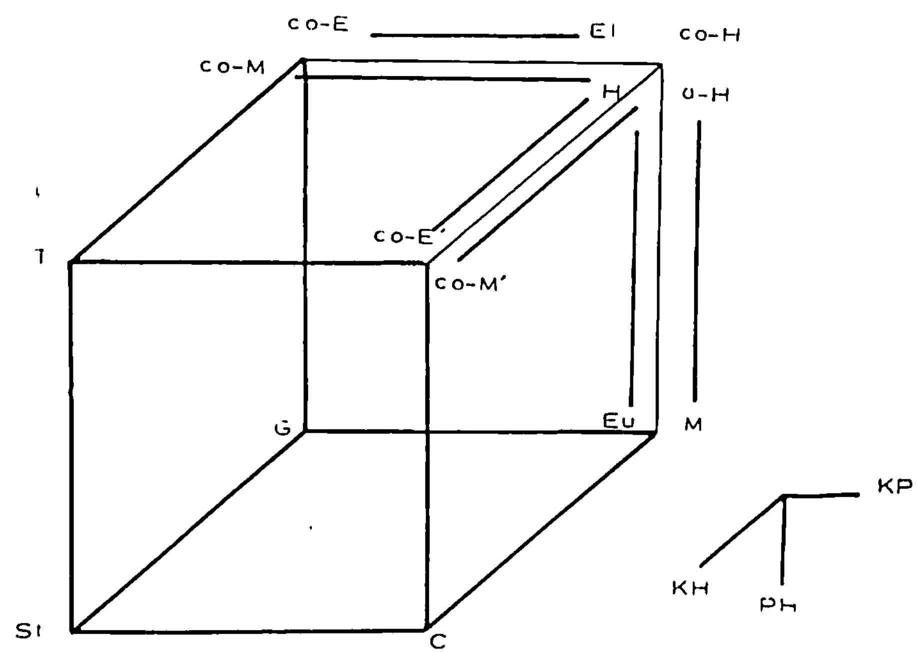

Fig. 4.

There is an abstract principle of relativity (Levy-Leblond, 1976) which takes us to various "theories of relativity" depending on the different laws of composition of the transformation groups.

Briefly, each "theory of relativity" is formed according to a determined group structure of the changes of inertial systems of frame as a whole, groups which we call relativity or kinematic groups.

Two well-known examples of concrete realizations of this principle are the Galileo and the Poincaré groups, which represent, respectively, the classical mechanics and the relativistic one.

We are interested in unidimensional kinematics, so we will suppose that our kinematic group is a connected Lie group of three parameters and that it is a physical requirement which expresses the supposed continuity of space-time.

The group must be connected to exclude parity and time-reversal as automorphisms of the groups.

To obtain the different realizations we use some techniques of Lie algebras. We will call H, P, and K the infinitesimal generators of time and space-translations and the rotation respectively.

Knowing the results demonstrated in Bacry and Levy-Leblond (1968) and thinking of our unidimensional case, we find the classification of unidimensional kinematic groups shown in Table V.

3.1. Contractions of the Kinematic Groups. All the kinematic groups we have seen before can be related through a process of approximation known as group contraction.

For instance, the Galileo group is obtained through a contraction of the Poincaré group with $P \rightarrow \varepsilon P$ and $K \rightarrow \varepsilon K$, H remaining unaltered, replacing the Lie algebra, and considering the limit $\varepsilon \rightarrow 0$ of the Lie brackets.

The physical meaning is clear. As the factor ε is attached to P and K, the contracted group describes a situation in which the speeds are small and the space-translation is small too.

We can consider four kinds of physical contractions for any kinematic group. We will call them PK contraction, KH contraction, PH contraction, and the general one, according to the generators which are contracted.

Table V. Lie Algebras of the Unidimensional Kinematic Groups.

	Relative-time groups						Absolute-time groups				
	dS_+	dS_-	P	P'_+	P'_-	C	N_+	N_-	G	G'	St
$[P,H]$	K	$-K$	0	K	$-K$	0	K	$-K$	0	K	0
$[K,H]$	P	P	P	0	0	0	P	P	P	0	0
$[K,P]$	H	H	H	H	H	H	0	0	0	0	0

It is possible to build a lattice of contractions with all the kinematic groups to see graphically what the relationship for contraction between one kinematic group and another is.

We consider a cube where the direction x is the direction of the PK contraction, the direction y is the direction of the KH contraction and z that of the PH contraction.

Then the scheme is shown in Figure 5.

4. CONCLUDING REMARKS

Amongst the possible subalgebras which admit all the kinematic groups are the ones we will call kinematic subalgebras, which regulate the unidimensional motions (generated by K) and their spatial and temporal relations (generated by P and H).

In our case the same groups are already the same kinematic algebra because we have only considered unidimensional groups.

If we compare the algebra of each kinematic group with that of the geometries of Cayley and Klein amplified physically we can give a geometri-

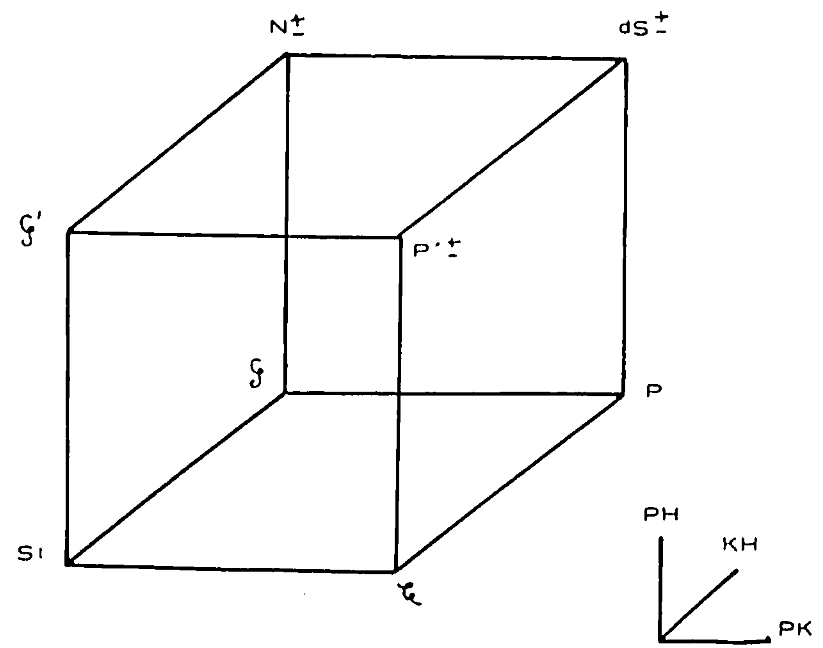

Fig. 5.

cal interpretation of each one of the different kinematic groups according to the different geometry they have associated.

Having done so the association is the following:

group	geometry	
$dS_+ \rightarrow$	doubly hyperbolic	(dH)
$dS_- \rightarrow$	co-Hyperbolic	(co-H)
$P \rightarrow$	Minkowski	(M)
$P'_+ \rightarrow$	co-Euclidean′	(co-E')
$P'_- \rightarrow$	co-Minkowski′	(co-M')
$\mathcal{C} \rightarrow$	Carroll	(C)
$N_+ \rightarrow$	co-Minkowski	(co-M)
$N_- \rightarrow$	co-euclidean	(co-E)
$\mathcal{G} \rightarrow$	Galilean	(G)
$\mathcal{G}' \rightarrow$	Translations	(T)
$St \rightarrow$	static	(St)

We see that the elliptic, Euclidean, and hyperbolic geometries do not appear, because they have groups which do not satisfy the third hypothesis of Bacry and Levy-Leblond (1968), that is, groups whose inertial transformations form a compact subgroup.

When this association is made, we can compare the schemes of contractions of the groups and geometries and we can see that they are equivalent. Thus we can conclude that the true reason for the scheme of contractions of the groups is hidden in the relations of contractions of the associated geometries.

In short, we can state that the reason for maintaining that, for example, the Galilean group is an approximation, understood as a contraction, of the Poincaré group is that the Galilean geometry is an approximation, understood also as a contraction, of the Minkowskian geometry; and likewise for all the other kinematic groups.

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