Journal of Sound and Vibration (1995) **185**(4), 734–736

COMMENTS ON THE HAMILTONIAN FORMULATION FOR LINEAR AND NON-LINEAR OSCILLATORS INCLUDING DISSIPATION

M. A. F. Sanjuán

Departamento de Física e Instalaciones Aplicadas, E.T.S. de Arquitectura, Universidad Politécnica de Madrid, 28040 *Madrid, Spain*

(*Received* 23 *December* 1994)

The Hamiltonian formalism for the damped harmonic oscillator has recently received some attention [1–4], in which a canonical transformation has been used in order to remove the damping term in the original term. The original idea goes back, as far as the author knows, to the article of Bateman [5], where he introduces the transformation in the Lagrangian formulation for the linear damped harmonic oscillator, while trying to prove that a linear dissipative system can be derived from a variational principle. Later, Havas [6] studied the range of application of the Lagrangian and Hamiltonian formalism. This was also used by Denman and Buch [7] in order to study the Hamilton–Jacobi equation and to analyze dissipative systems for its possible treatment in quantum mechanics.

In reference [4], Nagem and Sandri found, in the case in which the natural frequency is zero, that the energy E_T is a constant of motion and that the two quantities K_1 and K_2,

$$K_1 = \dot{x} + \gamma x = \mathrm{e}^{-\gamma t} p_x + \gamma x/2, \qquad K_2 = \dot{x}\,\mathrm{e}^{\gamma t} = p_x - (\gamma/2)x\,\mathrm{e}^{\gamma t}, \tag{1, 2}$$

are constants of motion. Using a convenient canonical transformation, Lemos [8] showed the existence of the constant of motion

$$f(x, \dot{x}, t) = \mathrm{e}^{\gamma t}(\dot{x}^2 + \omega_0^2 x^2 + \gamma x\dot{x}) \tag{3}$$

for the general linear damped harmonic oscillator of equation

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0, \tag{4}$$

from which K_1 and K_2 can be obtained as particular cases. On the other hand, a generalization of the transformation used in references [1, 4] can be found in reference [9], where a linear oscillator with time-dependent friction and frequency is considered. In relation to the geometry of the harmonic oscillator in phase space see reference [10].

Concerning the Hamiltonian formalism for non-linear oscillators with dissipation terms, Denman and Buch [7] gave the general expressions for the Lagrangian and the Hamiltonian. This was developed later by Steeb and Kunick [11] for a class of dissipative dynamical systems with limit cycle and chaotic behaviour and the explicit Lagrange and Hamilton functions were given. Among the non-linear oscillators considered were the Duffing and the simple pendulum. Suppose that we consider the equation of motion of a non-linear oscillator in the form

$$\ddot{x} + \gamma\dot{x} + \mathrm{d}V(x)/\mathrm{d}x = F \sin \omega t, \tag{5}$$

70022–460X/95/340734 + 03 $12.00/0

where F is the amplitude and ω is the frequency of the external perturbation, γ is the damping coefficient, $V(x)$ is the potential, and x represents the displacement from the equilibrium position. The Lagrange and Hamilton functions for this case are

$$L=e^{\gamma t}\left\{\frac{\dot{x}^2}{2}-V(x,t)\right\}, \qquad H=\tfrac{1}{2}p^2\,e^{-\gamma t}+e^{\gamma t}V(x,t), \tag{6, 7}$$

where the time-dependent potential is given by

$$V(x,t)=V(x)-Fx\sin\omega t. \tag{8}$$

From this general expression, if $V(x)=x^2/2$ and $F=0$, we obtain the Lagrangian and Hamiltonian for the harmonic oscillator, which clearly coincide with the ones given by Rollins and Shivamoggi [2], which are simpler than in reference [1], precisely for the reason stated there: the elimination of the gauge factor in the Lagrangian.

REFERENCES

1. R. NAGEM, B. A. RHODES and G. V. H. SANDRI 1991 *Journal of Sound and Vibration* **144**, 536–538. Hamiltonian mechanics of the damped harmonic oscillator.
2. D. K. ROLLINS and B. K. SHIVAMOGGI 1992 *Journal of Sound and Vibration* **154**, 551. Comments on "Hamiltonian mechanics of the damped oscillator".
3. R. J. NAGEM and G. V. H. SANDRI 1994 *Journal of Sound and Vibration* **154**, 552–555. Authors' reply.
4. R. NAGEM and G. V. H. SANDRI 1994 *Journal of Sound and Vibration* **169**, 270–275. Canonical transformations and phase trajectories of the damped harmonic oscillator.
5. H. BATEMAN 1931 *Physical Review* **38**, 815–819. On dissipative systems and related variational principles.
6. P. HAVAS 1951 *Physical Review* **83**, 224. On the range of application of the Lagrange and Hamilton formalisms.
7. H. H. DENMAN and L. H. BUCH 1973 *Journal of Mathematical Physics* **14**, 326–329. Solution of the Hamilton-Jacobi equation for certain dissipative classical mechanical systems.
8. N. A. LEMOS 1979 *American Journal of Physics* **47**, 857–858. Canonical approach to the damped harmonic oscillator.
9. I. A. PEDROSA 1987 *Journal of Mathematical Physics* **28**, 2662–2664. Canonical transformations and exact invariant for dissipative systems.
10. K. R. MEYER 1990 *American Mathematical Monthly* **97**, 457–465. The geometry of harmonic oscillators.
11. W.-H. STEEB and A. KUNICK 1982 *Physical Review* **A25**, 2889–2892. Lagrange functions of a class of dynamical systems with limit-cycle and chaotic behavior.

AUTHORS' REPLY

R. J. NAGEM AND G. V. H. SANDRI

Department of Aerospace and Mechanical Engineering, *Boston University*, 110 *Cummington Street*, *Boston*, *Massachusetts* 02215, *U.S.A.*

(*Received* 20 *February* 1995)

We thank Dr Sanjuán for his interesting and enlightening comments. In addition to the results for ordinary differential equations which are listed by Dr Sanjuán, extensions to partial differential equations are also possible. For example, the equation

$$\partial^2\varphi/\partial t^2+2\gamma\partial\varphi/\partial t-c_0^2\partial^2\varphi/\partial x^2=0 \tag{1}$$

may be transformed by the change of variables

$$\varphi(x,t)=\mathrm{e}^{-\gamma t}\phi(x,t) \tag{2}$$

into the equation

$$\partial^2\phi/\partial t^2-c_0^2\partial^2\phi/\partial x^2-\gamma^2\phi=0. \tag{3}$$

Thus the physical problem of small transverse vibrations of a stretched string with viscous damping may be transformed into the problem of small transverse vibrations of an undamped stretched string on an elastic foundation with a negative spring constant. The Lagrangian density corresponding to equation (3) is

$$L_\phi=\tfrac{1}{2}(\partial\phi/\partial t)^2-\tfrac{1}{2}c_0^2(\partial\phi/\partial x)^2+\tfrac{1}{2}\gamma^2\phi^2, \tag{4}$$

and equation (3) implies the local energy conservation equation

$$\frac{\partial}{\partial t}\left\{\tfrac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2+\tfrac{1}{2}c_0^2\left(\frac{\partial\phi}{\partial x}\right)^2-\tfrac{1}{2}\gamma^2\phi^2\right\}+\frac{\partial}{\partial x}\left\{-c_0^2\left(\frac{\partial\phi}{\partial x}\right)\left(\frac{\partial\phi}{\partial t}\right)\right\}=0 \tag{5}$$

and the local conservation of linear momentum equation

$$\frac{\partial}{\partial t}\left\{-c_0^2\left(\frac{\partial\phi}{\partial x}\right)\left(\frac{\partial\phi}{\partial t}\right)\right\}+\frac{\partial}{\partial x}\left\{\tfrac{1}{2}c_0^2\left(\frac{\partial\phi}{\partial t}\right)^2+\tfrac{1}{2}c_0^4\left(\frac{\partial\phi}{\partial\chi}\right)^2+\tfrac{1}{2}\gamma^2c_0^2\phi^2\right\}=0. \tag{6}$$

By using equation (2) to change back to the variable φ, the Lagrangian density for equation (1) may be written as

$$L_\varphi=\mathrm{e}^{2\gamma t}[\tfrac{1}{2}(\partial\varphi/\partial t)^2+\gamma\varphi\partial\varphi/\partial t+\gamma^2\varphi^2-\tfrac{1}{2}c_0^2(\partial\varphi/\partial x)^2]. \tag{7}$$

The corresponding local energy conservation for equation (1) is

$$\frac{\partial}{\partial t}\left\{\mathrm{e}^{2\gamma t}\left[\tfrac{1}{2}\left(\frac{\partial\varphi}{\partial t}\right)^2+\gamma\varphi\,\frac{\partial\varphi}{\partial t}+\tfrac{1}{2}c_0^2\left(\frac{\partial\varphi}{\partial x}\right)^2\right]\right\}+\frac{\partial}{\partial x}\left\{-c_0^2\,\mathrm{e}^{2\gamma t}\left[\left(\frac{\partial\varphi}{\partial x}\right)\left(\frac{\partial\varphi}{\partial t}\right)+\gamma\varphi\,\frac{\partial\varphi}{\partial x}\right]\right\}=0, \tag{8}$$

and the local conservation of linear momentum equation for equation (1) is

$$\frac{\partial}{\partial t}\left\{-c_0^2\,\mathrm{e}^{2\gamma t}\left[\left(\frac{\partial\varphi}{\partial x}\right)\left(\frac{\partial\varphi}{\partial t}\right)+\gamma\varphi\left(\frac{\partial\varphi}{\partial x}\right)\right]\right\}$$
$$+\frac{\partial}{\partial x}\left\{\mathrm{e}^{2\gamma t}\left[\tfrac{1}{2}c_0^2\left(\frac{\partial\varphi}{\partial t}\right)^2+\gamma c_0^2\varphi\left(\frac{\partial\varphi}{\partial t}\right)+\gamma^2c_0^2\varphi^2+\tfrac{1}{2}c_0^4\left(\frac{\partial\varphi}{\partial x}\right)^2\right]\right\}=0. \tag{9}$$

If equation (1) is written with more than one space dimension, it is also possible to derive a local conservation of angular momentum equation.