Indecomposable Continua and the Characterization of Strange Sets in Nonlinear Dynamics

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We discuss a topological property which we believe provides a useful conceptual characterization of a variety of strange sets occurring in nonlinear dynamics (e.g., strange attractors, fractal basin boundaries, and stable and unstable manifolds of chaotic saddles). Sets with this topological property are known as *indecomposable continua*. As an example, we give detailed results for the case of an indecomposable continuum that arises from the entrainment of dye advected by a fluid flowing past a cylinder. We show for this case that the indecomposable continuum persists in the presence of small noise. [S0031-9007(97)02530-1]

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Sets that are loosely called "strange" occur commonly in nonlinear dynamics. Examples are strange attractors of chaotic systems, fractal basin boundaries, the stable and unstable manifolds of chaotic scattering sets (or other chaotic transients), and strange nonchaotic attractors. The singular fine-scaled structure of such sets is most commonly characterized by saying that these sets are fractal. In this paper we wish to introduce a concept from topology with we believe provides another useful characterization of many (not all) sets that would commonly be called strange. In particular, we discuss the applicability of the concept of *indecomposable continua* [1] in nonlinear dynamics. We also provide a detailed example involving the indecomposable continuum arising from the entrainment of dye advected by a fluid flowing past a cylinder, and for that example we show that the existence of an indecomposable continuum persists in the presence of small noise. In formulating the latter result, the concept of indecomposable continuum is essential, because the usual approach to the noise-free problem is to show the presence of a fractal chaotic invariant set, and with noise there are no invariant sets.

A *continuum* is a compact, connected [2] set. It is called *decomposable* when it can be regarded as the union of two overlapping subcontinua. For example, the shaded area in Fig. 1(a) is a continuum. It is decomposable because we can divide it into two subsets by the line shown in the figure, and each of the two subsets are continua (they overlap if we take the two subsets to each include the dividing line). Other examples of decomposable continua are a line segment, the three-dimensional volume on a solid cube, and the surface of the cube. On the other hand, every indecomposable continuum has the strange property that if you attempt to divide it into two parts, then each of those parts has an uncountable number of connected pieces. That is, the division causes the original object to "shatter" into an infinite number of pieces [3].

As an example of an indecomposable continuum, consider the strange attractor in Fig. 1(b). This set is clearly compact and connected [2], and is hence a continuum. It is also indecomposable. For example, consider the component subsets lying inside and outside the dividing oval shown in the figure. Noting the Cantor structure of the attractor transverse to the apparent smooth variation along the unstable direction, we surmise that the component subsets contain an uncountable number of disjoint pieces (corresponding to the uncountable number of disjoint points in a Cantor set). Note that, by definition,

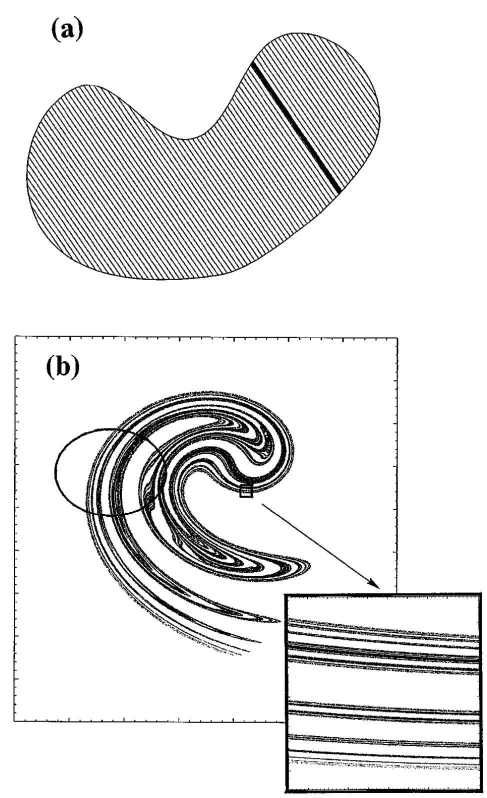

FIG. 1. (a) Example of a decomposable continuum. (b) A strange attractor exemplifying an indecomposable continuum which arises from the Ikeda map $z_{n+1} = 1 + 0.9 z_n \exp\{i[0.4 - 6(1 + |z_n|^2)^{-1}]\}$, $z = x + iy$.

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this applies for any smooth division of an indecomposable continuum. In particular, assume that we construct a tiny square about a point on the attractor in Fig. 1(b), and take the division of the attractor to be the inside and outside of the tiny square. Then the tiny square must contain an infinite number of disconnected components. Thus we always see structure as we examine stronger and stronger magnifications about the point [see the inset in Fig. 1(b)]. Hence, the indecomposable continuum property implies structure on an arbitrarily small scale.

As another example, which we pursue further in this paper, we consider a two-dimensional incompressible fluid flowing past a cylinder. As the flow velocity (i.e., Reynolds number) increases, it is well known that the steady flow becomes unstable, and the flow becomes time periodic, $\mathbf{v}(\xi, t) = \mathbf{v}(\xi, t + T)$, where $\xi = (x, y)$, T denotes the period, and $\mathbf{v}$ is the Eulerian fluid velocity. In this time periodic regime, vortices are alternatively shed from either side of the cylinder and advected downstream. This situation has been extensively considered from the dynamical systems point of view [4–6]. In these works evidence has been presented that the dynamics of fluid trajectories given by $d\xi/dt = \mathbf{v}(\xi, t)$ yields a chaotic invariant set for the associated time-T map. Furthermore, it has also been noted that this should lead to fractal properties of tracer particles originally placed upstream from the chaotic invariant set.

An example is shown in Fig. 2 with the flow specified by a stream function $\psi(x, y, t)$. Here $v_x = -\partial\psi/\partial y$ and $v_y = \partial\psi/\partial x$, and for $\psi(x, y, t)$ we use a form given in Ref. [4] to model the previously described periodic vortex shedding flow. Imagine that at $t = 0$ the fluid in the region, $x < 0$, $y > 0$, is dyed [we take $(x, y) = (0, 0)$ to be the center of the cylinder] and that the flow is in the positive x direction. Figure 2(a) shows a depiction of a snapshot of the fluid taken at a later time nT, where n is an integer. The dyed fluid is shown as black. As n is increased the boundary of the black region asymptotes to a set which includes the fractal unstable manifold of a chaotic invariant set. This set is also an indecomposable continuum [7]. The focus of the present Letter is the question of what happens when the periodic flow is perturbed by noise. That is, the fluid velocity is now $\mathbf{v} = \mathbf{v}_0 + \delta\mathbf{v}$, where $\mathbf{v}_0$ is the original time periodic flow and $\delta\mathbf{v}$ is a small nonperiodic fluctuating perturbation. If one integrates the trajectory equation, $d\xi/dt = \mathbf{v}$, forward from time nT to time $(n + 1)T$, then a map which is explicitly dependent on n results, $\xi_{n+1} = \mathbf{F}_n(\xi_n)$, $\xi_n = \xi(nT)$. In the case of a time periodic flow, $\delta\mathbf{v} = 0$, the map is n independent, $\xi_{n+1} = \mathbf{F}(\xi_n)$. We shall show that fractal properties, as in Fig. 2(a), persist for small $\delta\mathbf{v}$. (In this case the map deviation from the purely periodic case, $\mathbf{F}_n(\xi) - \mathbf{F}(\xi)$, is small and varies irregularly with time n; see also [8].)

In particular, assume that a finite cloud of dye is initialized upstream from the cylinder. As time proceeds, it is swept toward the cylinder, and most, not all, of it is subsequently advected downstream from the cylinder. A small amount (which continually decreases with time) is entrained by the flow in the region of the cylinder. As time increases the pattern formed by this entrained dye asymptotes to a fractal indecomposable continuum which we call the "entrainment set." Note that, in the presence of noise $\delta\mathbf{v}$ this set is not invariant but "jumps around" in a temporally irregular manner.

To begin, we first present an analysis of the case $\delta\mathbf{v} = 0$ with $\mathbf{v}_0$ specified by the time periodic stream function used in Ref. [4] and Fig. 2(a). There is a symmetry of the time periodic flow inherent in our description of alternate shedding of vortices from the top and bottom of the cylinder, namely, $\psi(x, y, t) = \psi(x, -y, t + T/2)$. This has the consequence that the map $\mathbf{F}$ has a "square root" $\mathbf{G}$, that is, $\mathbf{F} = \mathbf{G}^2$. To show this, we write $\mathbf{F} = \mathbf{H}_b\mathbf{H}_a$, where $\mathbf{H}_a(\mathbf{H}_b)$ is the map obtained via integration of $d\xi/dt = \mathbf{v}$ from time $t = nT$ to time $t = (n + \frac{1}{2})T$ [from $t = (n + \frac{1}{2})T$ to $t = (n + 1)T$]. If we write $\mathbf{H}_a$ as $\mathbf{H}_a(x, y) = [p(x, y), q(x, y)]$, then, by the symmetry, $\mathbf{H}_b(x, y) = [p(x, -y), -q(x, -y)]$. Hence, $\mathbf{F}(x, y) = [p(p(x, y), -q(x, y)), -q(p(x, y), -q(x, y))]$, which can be expressed as $\mathbf{F} = \mathbf{G}^2$ with $\mathbf{G}(x, y) = [p(x, y), -q(x, y)]$. We can numerically generate the map $\mathbf{G}$ by integrating the flow $d\xi/dt = \mathbf{v}$ from $t = nT$ to $t = (n + \frac{1}{2})T$ and then reflecting the resulting position about the x axis, $y \to -y$. Doing this, we numerically find that the map $\mathbf{G}$ has a horseshoe. That is, there is a rectangle Q such that $\mathbf{G}(Q)$ horizontally stretches Q completely across Q at least two times (in this case, three times), and $\mathbf{G}(Q)$ does not intersect the horizontal sides of Q. Using a theorem of Barge [9], the existence of a horseshoe implies that there is an indecomposable

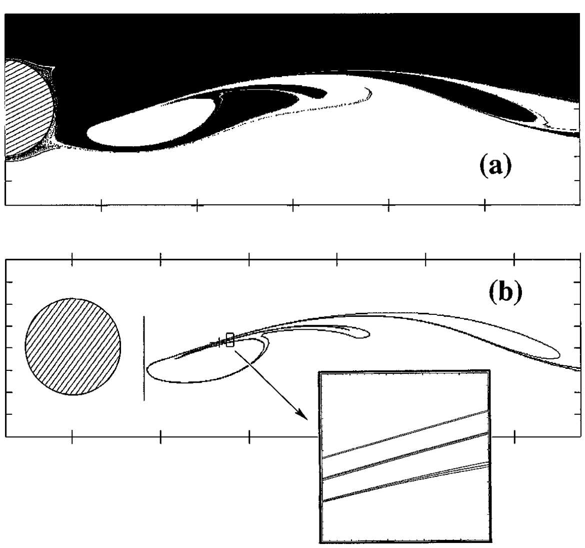

FIG. 2. (a) The fluid at $n = 3$. Particles of dye whose initial positions are in $x < 0$, $y > 0$ are marked black and the rest are marked white. (b) The vertical line of dye is advected by the fluid and accumulates on the unstable manifold of a chaotic saddle, which is also an indecomposable continuum. The inset in (b) shows a blowup of the small rectangle in (b). The cylinder is shown crosshatched.

continuum for the map $\mathbf{G}$, and hence also for the map $\mathbf{F}$. We numerically obtain a picture of the indecomposable continuum resulting from this horseshoe as follows. We find that each point on the vertical line segment shown in Fig. 2(b) is mapped strictly to the right of the line segment and that the segment is to the left of Q. Thinking of the line segment as a line of dye, we take it forward many iterates of $\mathbf{F}$ and obtain the entrainment set shown in Fig. 2(b). The fine-scaled structure is verified for this entrainment set by the blowup shown in the inset.

We now discuss the case where the original time periodic flow $\mathbf{v}_0$ is perturbed by the small noise $\delta\mathbf{v}$. We assume that $|\mathbf{F}_n(\xi) - \mathbf{F}(\xi)| < \epsilon$ and $|\mathbf{DF}_n(\xi) - \mathbf{DF}(\xi)| < \epsilon$ for all n and ξ, and we call ϵ the "noise level." In particular, since $\mathbf{F}$ is a horseshoe map on the region Q, each of the perturbed maps $\mathbf{F}_n$ are individually horseshoe maps on Q for sufficiently small ϵ. Below, we shall state and discuss a rigorous result applicable when this hypothesis for small ϵ is satisfied. Proofs will be provided elsewhere [6].

Let $\tilde{S}^+(x_0)$ be the set of ξ such that the noisy trajectory points (generated by the sequence $\mathbf{F}_n$ starting from ξ at time $n = 0$ remain to the right of x_0 for all positive and negative time n [7].

Theorem.—For every x_0 *to the left of* Q, $\tilde{S}^+(x_0)$ *contains an indecomposable continuum.*

In the case $\epsilon = 0$, the set $\tilde{S}^+(x_0)$ contains the unstable manifold of the chaotic invariant set, and for either $\epsilon = 0$ or $\epsilon \neq 0$ the set $\tilde{S}^+(x_0)$ can be identified with an entrainment set. We can give some of the key features leading to this theorem by considering a simpler, but essentially equivalent, problem. In particular, we consider a sequence of maps $\mathbf{M}_n$ (analogous to our sequence $\mathbf{F}_n$). The $\mathbf{M}_n$ are random, but we choose them all to individually have the property that they map the stadium-shaped region $D = A \bigcup B \bigcup C$ shown in Fig. 3(a) to a region of the form shown crosshatched in Fig. 3(b). We emphasize that the exact location and shape of the crosshatched region in Fig. 3(b) is different for each n, but the general property that $\mathbf{M}_n(C)$ stretches twice across C, and that $\mathbf{M}_n(A)$ and $\mathbf{M}_n(B)$ are both located in A, holds for each $\mathbf{M}_n$. We now introduce the notation $\widetilde{\mathbf{M}}^m(\xi) = \mathbf{M}_0(\mathbf{M}_{-1}(\mathbf{M}_{-2}(\ldots\mathbf{M}_{-(m-1)}(\xi))\ldots))$. That is, we look at the trajectory starting from $\xi_{-(m-1)}$ at the negative time $-(m-1)$ and ask, "'What is the trajectory location at time zero ξ_0?" (In the usual case where all the maps are equal, $\widetilde{\mathbf{M}}^m$ reduces to $\mathbf{M}^m$, the mth iterate of $\mathbf{M}$.) The action of $\widetilde{\mathbf{M}}^m$ on the region $D = A \bigcup B \bigcup C$ is illustrated in Fig. 3(c) for $m = 1$ and $m = 2$. In particular, $\widetilde{\mathbf{M}}^1 \equiv \mathbf{M}_0$ maps D to a region contained within D, and $\widetilde{\mathbf{M}}^2$ maps D to a region contained within $\widetilde{\mathbf{M}}^1(D)$. In general, $\widetilde{\mathbf{M}}^{m+1}(D) \subset \widetilde{\mathbf{M}}^m(D)$. Each of the $\widetilde{\mathbf{M}}^m(D)$ is a compact continuum. We now consider the set $\Sigma = \bigcap_{m=1}^{\infty} \widetilde{\mathbf{M}}^m(D)$. An elementary theorem from topology tells us that the intersection of nested continua is itself a continuum. Thus Σ is a continuum. In particular, another theorem from topology states that any nested intersection of compact sets is nonempty and compact. Finally, we can surmise from the fractal structure evident from the process of successively intersecting the sets $\widetilde{\mathbf{M}}^m(D)$ that Σ is indecomposable. Thus Σ is an indecomposable continuum.

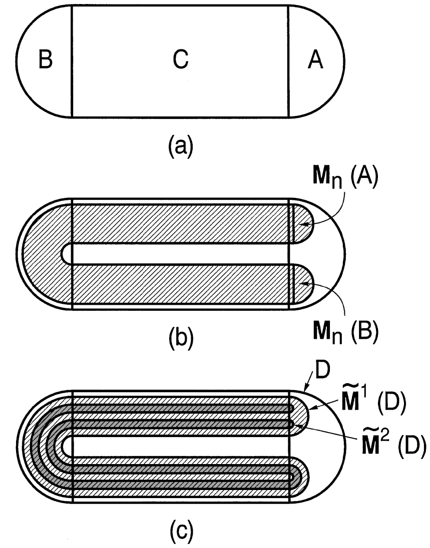

FIG. 3. Horseshoe illustration for the random map sequence $\mathbf{M}_n$.

We now discuss the applicability of the model sequence $\mathbf{M}_n$ illustrated in Fig. 3 to the sequence $\mathbf{F}_n$ realized by our noisy flow [10]. The most important difference between $\mathbf{M}_n$ and $\mathbf{F}_n$ arises because the maps $\mathbf{F}_n$ are area preserving. On the other hand, we see from Fig. 3(b) that $\mathbf{M}_n$ maps A into a subset of A. Thus the maps $\mathbf{M}_n$ are area decreasing on A. While we do not give the argument here, it can be shown that the construction in Fig. 3 can be recast in such a way as to make it topologically equivalent to the area preserving case. (This is done by allowing the regions A and B to extend to $\xi = \infty$ [7] and to have infinite area. Thus $\mathbf{F}_n(A)$ and $\mathbf{F}_n(B)$, while of infinite area, can both be contained within A.)

In conclusion, we have explored a numerical example of a fluid past a cylinder. Our goal has been to study a fluid flow which is temporally periodic flow plus a time varying perturbation. Under such circumstances, no bounded invariant sets are preserved. We show that it is nonetheless possible to discuss fractal sets that remain. These are indecomposable continua which correspond to physically observable remnants of dye introduced earlier into the fluid. More generally, we suggest that the concept of indecomposable continua may be useful for characterizing a variety of types of sets commonly occurring in chaotic dynamics.

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[1] J. Kennedy, in *Continua*, edited by H. Cook, W. T. Ingram, K. T. Kuperberg, A. Lelek, and P. Minc (Marcel Dekker, New York, 1995), pp. 103–126.

[2] A closed set is *connected* if it cannot be written as the disjoint union of two nonempty closed subsets. A set E is *compact* if every infinite subset of E has a limit point in E.

[3] More precisely, if K is an indecomposable continuum set in R^2, then $K \bigcap T$ has uncountably many connected pieces for any closed set T, such that (i) T contains a disc, (ii) K intersects the interior of the disc, and (iii) the set K is not entirely contained within T.

[4] C. Jung and E. Ziemniak, J. Phys. A **25**, 3929 (1992); C. Jung, T. Tél, and E. Ziemniak, Chaos **3**, 555 (1993); A. Péntek, Z. Toroczkai, T. Tél, C. Grebogi, and J. A. Yorke, Phys. Rev. E **51**, 4076 (1995).

[5] An experiment has also recently been carried out on such flow [J. C. Sommerer (to be published)].

[6] M. A. F. Sanjuán, J. Kennedy, C. Grebogi, and J. A. Yorke, 1996) (to be published).

[7] For the definition of an indecomposable continuum, in this case we add the point ∞ to R^2 to form the compact space $\overline{R}^2 = R^2 \bigcup \{\infty\}$. In particular, the continua for our fluid flow include the point at infinity.

[8] Maps varying irregularly with the iteraction number n have been used to model temporally irregular fluid flows, by for example, L. Yu *et al.* [Phys. Rev. Lett. **65**, 2935 (1990)], J. C. Sommerer and E. Ott [Science **259**, 335 (1993)], and T. M. Antonsen *et al.* [Phys. Rev. Lett. **75**, 1751 (1995); *ibid.* 3438 (1995)]. In addition, in work complementary to that reported in our paper, J. Jacobs *et al.* (to be published) has used random maps to numerically investigate entrainment sets of temporally irregular open flows where the irregularity is not small. They numerically demonstrate the existence of multifractal entrainment sets and relate their D_q spectra to the distribution of finite Lyapunov exponents.

[9] M. Barge, Proc. AMS **101**, 541 (1987).

[10] Since $\mathbf{G}$ maps Q (analogous to C) three times across Q, the map $\mathbf{F} = \mathbf{G}^2$ (and hence the maps $\mathbf{F}_n$ for ϵ sufficiently small) maps Q nine times across Q. In contrast, $\mathbf{M}_n$ maps C twice across C. This, however, does not introduce an essential change in the argument.

[11] H. E. Nusse and J. A. Yorke, *Dynamics: Numerical Explorations* (Springer-Verlag, New York, 1994).