International Journal of Bifurcation and Chaos, Vol. 10, No. 9 (2000) 2257–2267

ANALYTICAL ESTIMATES OF THE EFFECT OF NONLINEAR DAMPING IN SOME NONLINEAR OSCILLATORS

JOSÉ L. TRUEBA, JOAQUÍN RAMS and MIGUEL A. F. SANJUÁN
Departamento de Ciencias Experimentales e Ingeniería,
Escuela Superior de Ciencias Experimentales y Tecnología,
Universidad Rey Juan Carlos, Tulipán s/n, 28933 Móstoles, Madrid, Spain

Received July 19, 1999; Revised June 4, 2000

This paper reports on the effect of nonlinear damping on certain nonlinear oscillators, where analytical estimates provided by the Melnikov theory are obtained. We assume general nonlinear damping terms proportional to the power of velocity. General and useful expressions for the nonlinearly damped Duffing oscillator and for the nonlinearly damped simple pendulum are computed. They provide the critical parameters in terms of the damping coefficient and damping exponent, that is, the power of the velocity, for which complicated behavior is expected. We also consider generalized nonlinear damped systems, which may contain several nonlinear damping terms. Using the idea of Melnikov equivalence, we show that the effect of nonlinear dissipation can be equivalent to a linearly damped nonlinear oscillator with a modified damping coefficient.

1. Introduction

Dissipation plays an important role in modeling nonlinear dynamical systems. In spite of its complicated physical origin, linear dissipative forces depending on the velocity are commonly used. However the frictional or drag forces which describe the motion of an object through a fluid or gas are usually very complex, and power-law dissipative terms in the velocity are used.

The mathematical form of the drag forces is usually experimentally determined with the use of wind tunnels or water tanks. Among the possible models, one of the simplest empirical mathematical model of the drag force is taken as $f(v) \propto v|v|^{p-1}$, where v represents the velocity of the object. Phenomenological models accounting for nonlinear dissipation seem to be natural and have been considered in many applied sciences, such as ship dynamics, vibration engineering, the study of the dynamics of galaxies, and the motion of projectiles [Bikdash *et al.*, 1994; Falzarano *et al.*, 1992; Huilgol *et al.*, 1995; Ravindra & Mallik, 1994a, 1994b; Sanjuán, 1999; Pfenniger & Norman, 1990; De Mestre, 1990].

The effect of some physical parameters, such as wave amplitude and nonlinear damping on the transient motions and the global system behavior of a capsizing ship was investigated by Falzarano *et al.* [1992]. In particular they use a model of a double-well and single-well Duffing oscillator with a linear plus a quadratic viscous damping term. A similar model for the motion of a mass hanging from a overhead crane, using a pendulum with a quadratic friction term was used by Huilgol *et al.* [1995]. In a recent work by Litak *et al.* [1999] the dynamical behavior of the Froude pendulum is analyzed, which may be understood as a pendulum with linear plus cubic damping terms, in particular the nonlinear damping term is of Rayleigh type. The stability of a nonlinearly damped hard Duffing oscillator has been analyzed in [Ravindra & Mallik, 1994a]. Furthermore, the role of nonlinear dissipation in soft Duffing oscillators has received some attention in [Ravindra & Mallik, 1994b], with the conclusion that nonlinear damping terms affects notably the onset of the period-doubling route to chaos. However, they underestimated the effect of nonlinear damping terms on the structure of the chaotic attractors. A simple nonlinear oscillator with a quadratic friction term, that does not change its sign depending on the sign of the velocity, in relation to the dynamics of avalanches in real sandpiles was considered by Linz [1995]. A model of a parametrically forced pendulum with a quadratic damping term is used by Pfenniger and Norman [1990] to model dissipation in barred galaxies, that is, ovally distorted galaxies in the region where their rotation curve is steep. Xie and Hue [1995] studied the dynamics of a double-well Duffing oscillator with a nonlinear damping term, where a parametric perturbation plays the role of the damping coefficient. Awrejcewicz and Holicke [1999] analyzed a nonlinear oscillator with a different type of nonlinear damping to model dry friction and stick-slip chaotic oscillations. Also related to dry friction, numerical investigations have been done by Lim and Chen [1998]. The role of nonlinear dissipation on some properties of the dynamics of the universal escape oscillator, such as the threshold of period-doubling bifurcations, fractal basin boundaries and the destruction of basins of attraction was considered by Sanjuán [1999].

In the present work we study the effect of the nonlinear dissipation on the dynamics of certain nonlinear oscillators. One possible way of classifying nonlinear oscillators is according to its behavior to external driving. Those oscillators which possess a stable limit cycle without external driving are called self-excited, and those which tend to rest when not driven are called strictly dissipative oscillators. One type of self-excited oscillator is the Van der Pol oscillator which possesses a nonlinear damping term. However, in all our analysis we will assume strictly dissipative forces. In particular, we analyze the double-well Duffing oscillator and the simple pendulum. We use the Melnikov method to evaluate the critical parameters for which complicated behavior is expected. This type of analysis has been done by different authors [Bikdash *et al.*, 1994; Falzarano *et al.*, 1992; Huilgol *et al.*, 1995, Ravindra & Mallik, 1994b; Sanjuán, 1999; Litak *et al.*, 1999]. Our analysis consider generalized nonlinear damping terms including every positive power of the velocity, and the Melnikov technique, applied to our models, allows us to obtain generalized expressions for the critical parameters for which fractal basin boundaries are created [Moon & Li, 1985], which may include as particular cases most of the analyses previously done. In this respect, our computations are indeed useful, since they can automatically provide the right results of previous works [Falzarano *et al.*, 1992; Huilgol *et al.*, 1995; Ravindra & Mallik, 1994b; Litak *et al.*, 1999]. Moreover, using the equivalence criterium introduced by Bikdash *et al.* [1994], we have developed the concept of Melnikov-equivalent damped oscillators. With it, we have shown that the introduction of a nonlinear damping term in a previously linearly damped oscillator may be equivalent, in the sense of Melnikov analysis, to a shift in the coefficient of the linear damping term.

2. The Nonlinearly Damped Double-Well Duffing Oscillator

One of the models of nonlinear oscillators that traditionally has received much attention in the literature is probably the Duffing oscillator. Besides its multiple applications in many different fields, it constitutes also a paradigm in the study of nonlinear oscillations where many new ideas can be tested. In this context we want to use it to check the effect on its dynamics of introducing nonlinear damping terms.

We consider the following equation of motion for the driven double-well Duffing oscillator with a nonlinear damping term

$$\ddot{x} + \alpha_p \dot{x}|\dot{x}|^{p-1} - x + x^3 = F \cos \omega t \,, \qquad (1)$$

where F is the amplitude and ω the frequency of the external perturbation. The nonlinear damping term is taken to be proportional to the power of the velocity, in the form $\alpha_p \dot{x}|\dot{x}|^{p-1}$, where $p \geq 1$ is the damping exponent, and α_p is the corresponding damping coefficient. A similar model was used previously by Ravindra and Mallik [1994a, 1994b] in their work on the role of nonlinear damping on some soft Duffing oscillators and by Sanjuán [1999] in his study of the nonlinearly damped escape oscillator. In the previous model of the nonlinearly damped double-well Duffing oscillator, the case $p = 1$ corresponds to the linear viscous damping term, which is rather well known in the literature.

Our purpose is to analytically estimate how certain nonlinear damping terms affects the dynamics of the nonlinear oscillator. Numerical experiments show that changing the damping exponent has a considerable effect on the global pattern of bifurcations of the dynamical system under consideration. The Melnikov method is one of the few analytical tools to study the global behavior of the system and, in particular, it gives a procedure for analyzing and estimating when a chaotic behavior of a certain dynamical system is expected. In order to apply this technique and to carry out this study, we need to consider the external forcing and the dissipation as small perturbations to the Hamiltonian system $\ddot{x} - x + x^3 = 0$. A corresponding Melnikov analysis for the linearly damped double-well Duffing oscillator, $p = 1$ in our model, has been considered extensively and as a reference it may be found in [Wiggins, 1990; Nayfeh & Balachandran, 1995]. One basic step for the application of the method is the calculation of the fixed points in phase space of the unperturbed integrable system. This system has two elliptic fixed points located at $(\pm 1, 0)$ and a hyperbolic fixed point located at $(0, 0)$. Moreover we need to compute the homoclinic orbits, that is, the solutions of the unperturbed system starting and ending at the hyperbolic fixed point. For that we simply need to integrate the system for the initial conditions $(\pm\sqrt{2}, 0)$, and the results are given by

$$(x_{sx}^{\pm}(t),\, y_{sx}^{\pm}(t)) = (\pm\sqrt{2}\,\mathrm{sech}\, t,\, \mp\sqrt{2}\,\mathrm{sech}\, t\, \tanh\, t)\,, \tag{2}$$

where the signs refer to the right and left half planes. Both solutions determine the separatrix orbit, since it separates two types of orbits in phase space.

According to Melnikov theory we need the previous ingredients in order to calculate the Melnikov function, which is associated with each homoclinic orbit. This function is calculated through the expression

$$M^{\pm}(t_0,\, \omega,\, p) = -\alpha_p \int_{-\infty}^{+\infty} |y_{sx}^{\pm}(t)|^{p+1} dt + F \int_{-\infty}^{+\infty} y_{sx}^{\pm}(t)\, \cos\, \omega(t + t_0) dt\,. \tag{3}$$

After substitution of the homoclinic solutions, where the Melnikov function need to be evaluated, it appears as

$$M^{\pm}(t_0, \omega, p) = -2^{\frac{p+1}{2}} \alpha_p \int_{-\infty}^{+\infty} \mathrm{sech}^{p+1} t\, \tanh^{p+1} t\, dt \tag{4}$$

$$\mp \sqrt{2} F\, \sin\, \omega t_0 \int_{-\infty}^{+\infty} \mathrm{sech}\, t\, \tanh\, t\, \sin\, \omega t\, dt\,. \tag{5}$$

The evaluation of the last integrals appear in the Appendix [Eqs. (A.1) and (A.3)], therefore

$$M^{\pm}(t_0,\, \omega,\, p) = -\alpha_p 2^{\frac{p+1}{2}} B\left(\frac{p+2}{2}\,,\, \frac{p+1}{2}\right) \pm F\sqrt{2}\pi\omega\, \mathrm{sech}\left[\frac{\pi\omega}{2}\right] \sin\, \omega t_0\,. \tag{6}$$

In this equation appears the function $B(r,\, s)$, which is the Euler Beta function and it can be easily evaluated in terms of the Euler Gamma function [Abramowitz & Stegun, 1970]. These functions are easily evaluated with the help of software packages such as *Maple* or *Mathematica*.

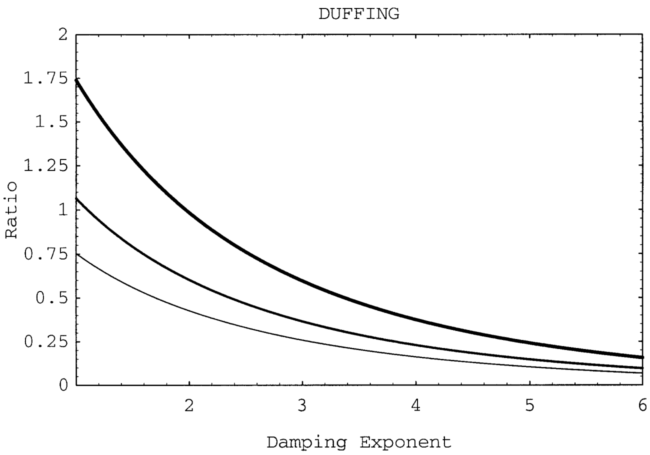
According to Melnikov theory, this function is related to the distance between the stable and the unstable manifolds associated with the hyperbolic fixed point, when destroyed by the perturbation. When this function has simple zeros, it implies that there is a critical parameter F_{cp} corresponding to the external forcing, for which homoclinic tangles intersect. For a certain frequency ω this critical parameter depends on the damping exponent p and the damping coefficient α_p. This critical parameter may be written as

$$F_{\mathrm{cp}} = \alpha_p \frac{2^{\frac{p}{2}}}{\pi\omega} B\left(\frac{p+2}{2},\, \frac{p+1}{2}\right) \cosh\left[\frac{\pi\omega}{2}\right]\,. \tag{7}$$

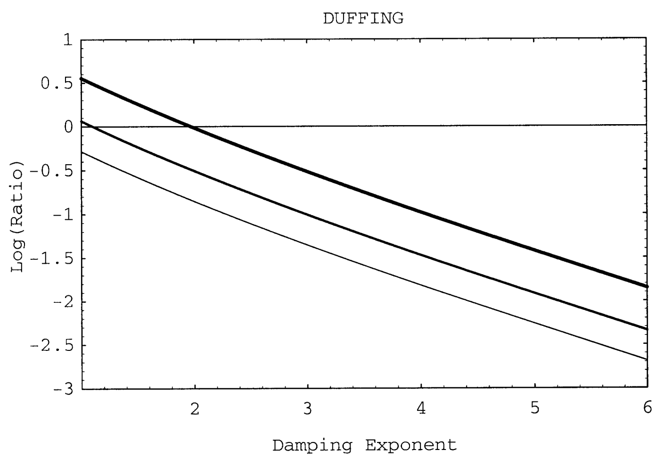
We define the function $D(\omega,\, p) = F_{\mathrm{cp}}/\alpha_p$, as the ratio between the external forcing and the damping coefficient and then we have the following expression

$$D(\omega,\, p) = \frac{2^{\frac{p}{2}}}{\pi\omega} B\left(\frac{p+2}{2},\, \frac{p+1}{2}\right) \cosh\left[\frac{\pi\omega}{2}\right]\,. \tag{8}$$

This last function is the one that provides information about the effect of the nonlinear damping on the threshold of homoclinic chaos and the appearance of fractal boundaries, as proved by Moon and Li [1985]. Accordingly, given a set of parameters of the system we may know when it is expected to find chaotic behavior in its dynamics.

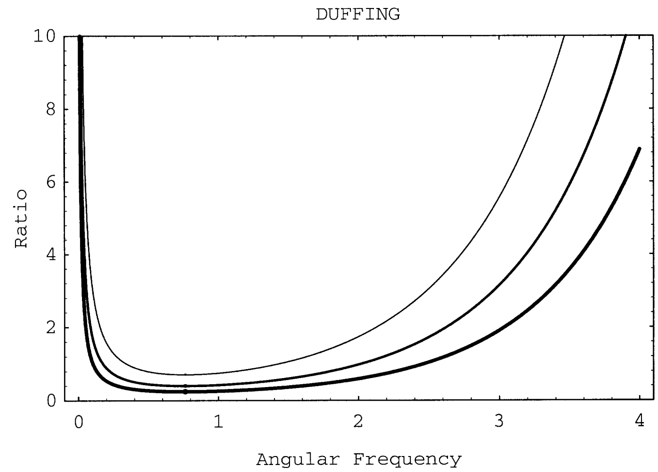


(a)

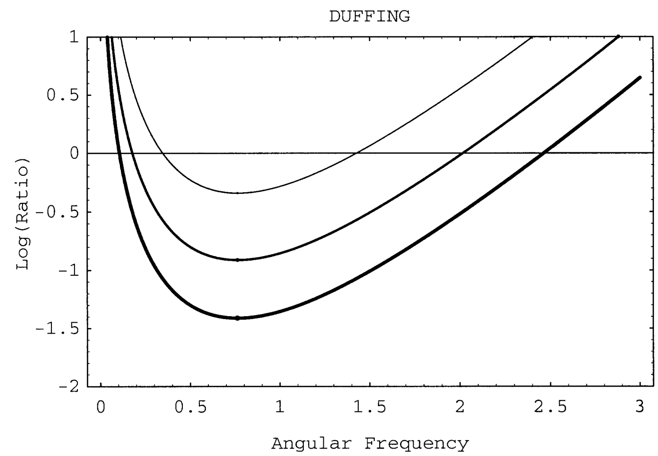


(b)

Fig. 1. Variation of the ratio $D(\omega,\, p)$ versus p corresponding to the Duffing oscillator $\ddot{x}+\alpha_p\dot{x}|\dot{x}|^{p-1}-x+x^3=F\,\cos\,\omega t$ for the angular frequencies $\omega=1$ (thin line), $\omega=1.5$ (medium line) and $\omega=2$ (thick line). (a) Standard scale; (b) semi-logarithmic scale.



(a)



(b)

Fig. 2. Variation of the ratio $D(\omega,\, p)$ versus ω corresponding to the Duffing oscillator $\ddot{x}+\alpha_p\dot{x}|\dot{x}|^{p-1}-x+x^3=F\,\cos\,\omega t$ for the damping exponents $p=1$ (thin line), $p=2$ (medium line) and $p=3$ (thick line). (a) Standard scale; (b) semi-logarithmic scale.

In order to have a visual information about this last result, we have plotted in Fig. 1(a) the dependence of the function $D(\omega,\, p)$ with respect to the damping exponent p for different values of the frequency of the external forcing, $\omega=1$, $\omega=1.5$ and $\omega=2$. One clear observation from this figure is that, for a fixed ω, the ratio of the external forcing to the damping coefficient, decreases when the damping exponent increases. This means that for higher values of the damping exponent, the critical parameter for which homoclinic chaos exist, decreases. In other words, with a smaller forcing parameter we may enter into a chaotic state. Moreover, this dependence is almost exponential, as is shown in Fig. 1(b), where a semi-logarithmic scale has been used. We have plotted in Fig. 2(a) the dependence of $D(\omega,\, p)$ with respect to the angular frequency ω for different values of the damping exponent, $p=1$, $p=2$ and $p=3$. This appears to be more clear in Fig. 2(b) where a semi-logarithmic scale is used. Consequently, what this figure shows is that for a fixed value of ω, the threshold for homoclinic chaos to occur decreases as the damping exponent p increases.

Numerical evidence shows that, for a fixed set of parameters, a period-doubling bifurcation is observed at a certain critical forcing amplitude.

Keeping fixed all the parameters, we have observed that the critical forcing amplitude decreases when the damping exponent increases.

3. The Nonlinearly Damped Simple Pendulum

Another paradigm in nonlinear dynamics is constituted by the simple pendulum. Analogously as in the case of the Duffing oscillator it has been used as a practical model for many different applications in science and engineering. As in the previous case, we consider in our model a damping term which is proportional to a power of the velocity, therefore the model equation for the nonlinearly damped simple pendulum is

$$\ddot{\theta} + \alpha_p \dot{\theta}|\dot{\theta}|^{p-1} + \sin\,\theta = F\,\cos\,\omega t\,. \tag{9}$$

For different values of the damping exponent p, a different dissipative force acts on the system. Numerical evidence also shows that different damping exponents produce a global pattern of bifurcations. Our aim here is to apply Melnikov method in order to obtain an analytical estimate of the effect of nonlinear damping on the dynamics of the simple pendulum. All we need is to compute the Melnikov function associated to this system. For that, we consider the external forcing and the nonlinear damping as a Hamiltonian perturbation to the simple pendulum with equation $\ddot{\theta} + \sin\,\theta = 0$. The Hamiltonian function for $(\theta,\,\dot{\theta}) \in [-\pi,\,\pi] \times R$ is

$$H(\theta, \dot{\theta}) = \frac{1}{2}\,\dot{\theta}^2 - \cos\,\theta\,. \tag{10}$$

The phase space of the pendulum is 2π-periodic in θ with hyperbolic saddles in $(\pm\pi,\,0)$ and an elliptic centre in $(0,\,0)$. This system has three kinds of solutions: Oscillations, rotations and a separatrix orbit. The solutions for the oscillating orbits can be expressed [Koch & Leven, 1985; Sanjuán, 1996] as

$$(\theta(t),\,\dot{\theta}(t)) = (2k\mathrm{cn}(\Omega t,\,k)\mathrm{dn}(\Omega t,\,k),\,2k\mathrm{cn}(\Omega t,\,k))\,, \tag{11}$$

where the functions $\mathrm{cn}(\Omega t,\,k)$ and $\mathrm{dn}(\Omega t,\,k)$ are Jacobi Elliptic Functions [Lawden, 1989; Abramowitz & Stegun, 1970] of frequency Ω and elliptic modulus k. It should be noted that sometimes the elliptic parameter m is used instead, where $m = k^2$. The solutions for the heteroclinic orbits are obtained by simply taking the limit $k \to 1$ for the elliptic modulus. The solutions are given by integrating $H(\theta, \dot{\theta}) = h$. For $h = 1$, then we have a pair of heteroclinic solutions given by

$$\theta_0^{\pm}(t) = \pm 2\,\arctan[\sinh\,t] = \pm 2\,\tanh\,t\,\mathrm{sech}\,t \tag{12}$$

$$\dot{\theta}_0^{\pm}(t) = \pm 2\,\mathrm{sech}\,t\,, \tag{13}$$

subjected to the initial conditions $(\theta_0^{\pm}(0),\,\dot{\theta}_0^{\pm}(0)) = (0,\,\pm 2)$. Thus the Melnikov function can be written as

$$\begin{aligned} M^{\pm}(t_0,\,\omega,\,p) = &-\alpha_p \int_{-\infty}^{+\infty} |\dot{\theta}_0^{\pm}(t)|^{p+1} dt \\ &\pm F\,\cos\,\omega t_0 \int_{-\infty}^{+\infty} \sin(\theta_0^{\pm}(t)) \\ &\times \dot{\theta}_0^{\pm}(t)\,\cos\,\omega t\,dt\,. \end{aligned} \tag{14}$$

After substitution of the heteroclinic solutions, the evaluation of these integrals [see Eqs. (A.1) and (A.2)] gives

$$\begin{aligned} M^{\pm}(t_0,\,\omega,\,p) = &-\alpha_p 2^{p+1} B\left(\frac{1}{2},\,\frac{p+1}{2}\right) \\ &\pm 2\pi F\,\mathrm{sech}\left(\frac{\pi\omega}{2}\right)\cos\,\omega t_0\,, \end{aligned} \tag{15}$$

where $B(r,\,s)$ is the Euler Beta function [Abramowitz & Stegun, 1970].

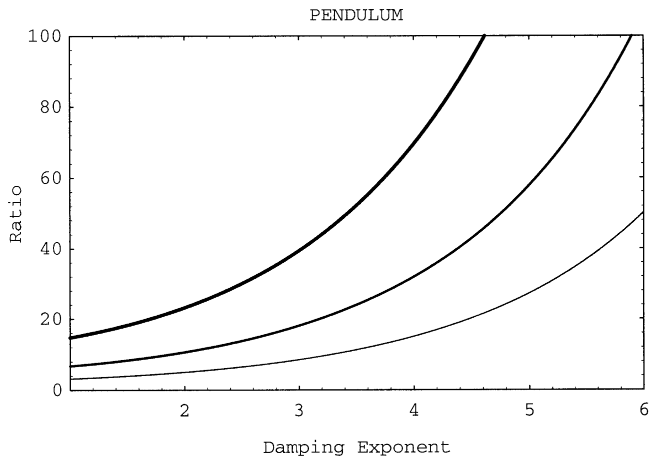
As a consequence of the Melnikov theory, the zeros of the Melnikov function provides the critical parameter F_{cp}, which depends on the damping exponent p, for which the stable and unstable manifolds associated with the hyperbolic saddle point intersect, is given by

$$F_{\mathrm{cp}} = \alpha_p\,\frac{2^p}{\pi}\,B\left(\frac{1}{2},\,\frac{p+1}{2}\right)\cosh\left[\frac{\pi\omega}{2}\right]. \tag{16}$$

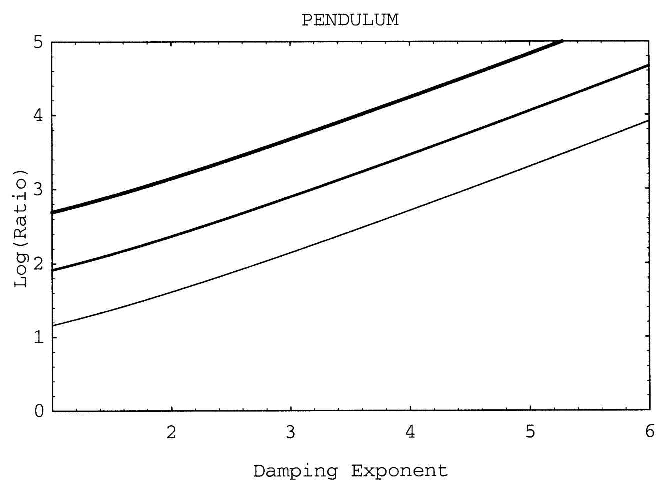
As in the previous case, we define the ratio of the forcing with respect to the damping as $P(\omega,\,p) = F_{\mathrm{cp}}/\alpha_p$, and then we have the following function for the pendulum

$$P(\omega,\,p) = \frac{2^p}{\pi}\,B\left(\frac{1}{2},\,\frac{p+1}{2}\right)\cosh\left[\frac{\pi\omega}{2}\right]. \tag{17}$$

Similarly to the Duffing oscillator, we visualize this information, given by the analytical methods, by plotting in Fig. 3(a) the dependence of $P(\omega,\,p)$ with respect to the damping exponent p, for different frequencies of the external perturbation, $\omega = 1$,

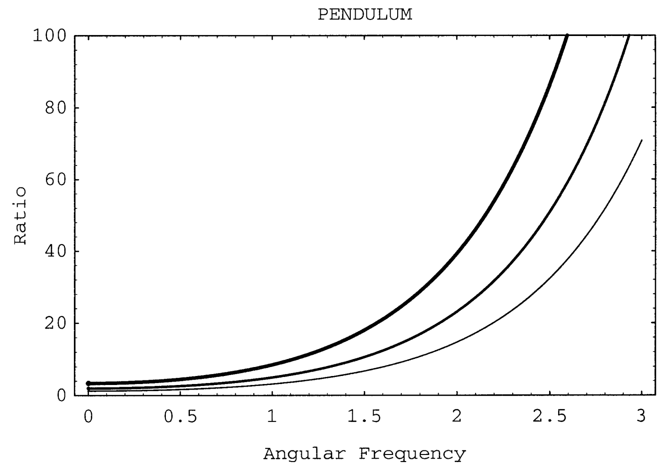


(a)

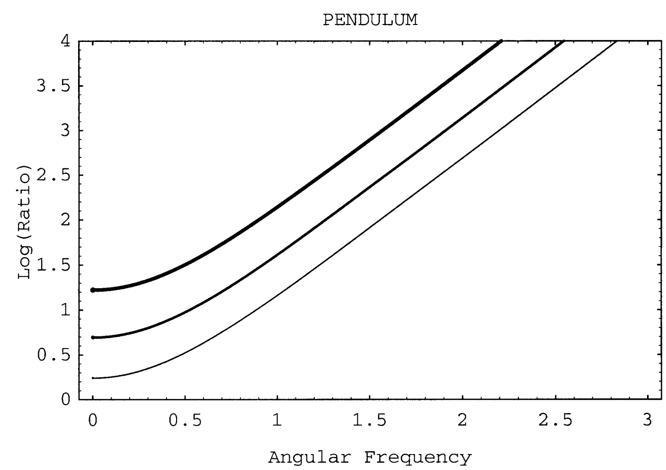


(b)

Fig. 3. Dependence of the ratio $P(p, \omega)$ versus p corresponding to the simple pendulum $\ddot{\theta} + \alpha_p \dot{\theta}|\dot{\theta}|^{p-1} + \sin\theta = F\cos\omega t$ for the angular frequencies $\omega = 1$ (thin line), $\omega = 1.5$ (medium line) and $\omega = 2$ (thick line). (a) Standard scale; (b) semi-logarithmic scale.



(a)



(b)

Fig. 4. Dependence of the ratio $P(p, \omega)$ versus ω corresponding to the simple pendulum $\ddot{\theta} + \alpha_p \dot{\theta}|\dot{\theta}|^{p-1} + \sin\theta = F\cos\omega t$ for the damping exponent $p = 1$ (thin line), $p = 2$ (medium line) and $p = 3$ (thick line). (a) Standard scale; (b) semi-logarithmic scale.

$\omega = 1.5$ and $\omega = 2$. Contrary to the case of the Duffing oscillator, this ratio increases when the damping exponent p increases. That is, for higher values of the damping exponent the threshold increases for the homoclinic chaos to occur. This idea appears to be more clearly illustrated in Fig. 3(b), where a semi-logarithmic scale is used. On the other hand, Fig. 4(a) shows the dependence of $P(\omega, p)$ with respect to the angular frequency for the damping exponents $p = 1$, $p = 2$ and $p = 3$. Similarly we use a semi-logarithmic scale in Fig. 4(b).

We have numerically simulated the system and, for a chosen set of parameters, we have observed that increasing the damping exponent produces the effect of decreasing the critical forcing amplitude for which a period-doubling bifurcation occurs.

4. Melnikov-Equivalent Damped Oscillators

4.1. *Duffing oscillator*

The previous analysis for the nonlinearly damped Duffing oscillator and for the nonlinearly damped pendulum can be easily generalized to the case in which more powers of velocity are included. In particular, and for the nonlinearly damped Duffing oscillator, we may find

$$\ddot{x} + \sum_{i=1}^{N} \alpha_{p_i} \dot{x} |\dot{x}|^{p_i - 1} - x + x^3 = F \cos \omega t \,, \qquad (18)$$

where N different nonlinear damping terms, with damping coefficients α_{p_i}, $p_i \geq 1$, are included. The application of Melnikov analysis to this system provides a Melnikov function that takes the form

$$M^{\pm}(t_0, \omega, p_i) = -\sum_{i=1}^{N} 2^{\frac{p_i+1}{2}} \alpha_{p_i} B\left(\frac{p_i+2}{2}, \frac{p_i+1}{2}\right) \pm \sqrt{2} F \pi\omega \operatorname{sech}\left(\frac{\pi\omega}{2}\right) \sin \omega t_0 \,, \qquad (19)$$

and consequently the critical parameter of the external perturbation is given by

$$F_{\mathrm{cp_i}} = \cosh\left(\frac{\pi\omega}{2}\right) \sum_{i=1}^{N} \frac{2^{p_i/2}}{\pi\omega} \alpha_{p_i} B\left(\frac{p_i+2}{2}, \frac{p_i+1}{2}\right). \qquad (20)$$

A damping term like the one we have just introduced can be called a *generalized nonlinear damping term*. On the other hand, the usual linear damping term which appears in the ordinary Duffing oscillator is called a *simple linear damping term*

$$\ddot{x} + \mu\dot{x} - x + x^3 = F \cos \omega t \,. \qquad (21)$$

Our aim here is to analyze what is the effect of the generalized nonlinear damping term with respect to the simple damping term. The analysis done above allows us to use the criterium of Melnikov equivalence, that is, *two generalized nonlinearly damped systems, having the same unperturbed differential equation, are Melnikov-equivalent if they have the same Melnikov function.* This criterium has been introduced in [Bikdash *et al.*, 1994] to study the influence of different damping models on the nonlinear roll dynamics of ships. To be precise, what we try to answer is when a nonlinear oscillator with a simple damping term is Melnikov-equivalent to a nonlinear oscillator with a generalized damping term.

The Melnikov function associated to the Duffing oscillator with a simple damping term is given by

$$M^{\pm}(t_0) = -\frac{4}{3}\mu \pm \sqrt{2} F \pi\omega \operatorname{sech}\left(\frac{\pi\omega}{2}\right) \sin \omega t_0 \,. \qquad (22)$$

Now we have two Duffing oscillators with different types of damping, one with a generalized nonlinear damping term and another one with a simple linear damping term. We have evaluated the corresponding Melnikov functions and according to the concept of Melnikov-equivalence that we have previously introduced, we conclude that both Melnikov functions are equal when the following condition holds

$$\mu = 3 \sum_{i=1}^{N} 2^{\frac{p_i-3}{2}} B\left(\frac{p_i+2}{2}, \frac{p_i+1}{2}\right) \alpha_{p_i} \,. \qquad (23)$$

The meaning of this expression is the following. Given a Duffing oscillator with a nonlinear generalized damping term, in which the set of parameters $(\alpha_{p_i}, F, \omega)$ are fixed, then there exists a Duffing oscillator with a simple linear damping term, which is Melnikov-equivalent under the following conditions: (i) The parameters F, ω are equal in both systems, and (ii) The parameter μ is related to the set of parameters α_{p_i} through Eq. (23). It is interesting to note from this result that the damping coefficient μ is a linear function of the damping coefficients α_{p_i}.

4.1.1. *Numerical simulation*

Next, we consider a particular case for which we attempt to apply this result and to numerically explore its dynamical behavior. We consider a nonlinear Duffing oscillator including a simple linear damping term plus a cubic damping term given by the following expression

$$\ddot{x} + \alpha_1\dot{x} + \alpha_3\dot{x}^3 - x + x^3 = F \cos \omega t \,. \qquad (24)$$

According to our previous result this system should be Melnikov-equivalent to the Duffing oscillator

$$\ddot{x} + \mu\dot{x} - x + x^3 = F \cos(\omega t) \,, \qquad (25)$$

if the damping coefficient μ satisfies the condition

$$\mu = \alpha_1 + \frac{12}{35}\alpha_3 \,. \qquad (26)$$

As an easy application of Melnikov theory, we can evaluate the critical parameter $F_{\mathrm{cp},1}$ of the generalized nonlinearly damped Duffing oscillator from the general expression (20), which is given by

$$F_{\mathrm{cp},1} = \frac{\sqrt{2}}{\pi\omega} \cosh\left(\frac{\pi\omega}{2}\right) \left(\frac{2}{3}\alpha_1 + \frac{8}{35}\alpha_3\right). \qquad (27)$$

Analogously, the critical parameter $F_{\text{cp},2}$ of the simple linearly damped Duffing oscillator is provided by

$$F_{\text{cp},2} = \frac{\sqrt{2}}{\pi\omega}\cosh\left(\frac{\pi\omega}{2}\right)\frac{2}{3}\mu\,. \tag{28}$$

The critical parameters that are obtained from both nonlinear oscillators are equal in the case where condition (26) is fulfilled.

We have numerically simulated the dynamics of the nonlinear oscillators using a fixed-step-size fourth-order Runge–Kutta numerical integration and its associated 2D Poincaré map. For our numerical computations we have fixed the parameters $\omega = 1$, $\alpha_1 = 0.1$, $\alpha_3 = 0.01$ in the system (24), that is, the one with a generalized nonlinear damping term. This gives an equivalent simple damped system (25) with a damping coefficient $\mu = 0.10343$ that we have computed by using condition (26), and the critical parameter is then given by $F_{\text{cp}} = F_{\text{cp},1} = F_{\text{cp},2} = 0.07788$.

For these conditions we have plotted in Fig. 5 the behavior of the stable and unstable manifolds of the system (24) with $\omega = 1$, $\alpha_1 = 0.1$, $\alpha_2 = 0.01$, and the equivalent system (25) with $\omega = 1$, $\mu = 0.10343$. Figure 5(a) refers to the situation in which the forcing amplitude is given by $F = 0.075$ for the generalized nonlinearly damped system (24), which is below the critical parameter $F_{\text{cp}} = 0.07788$. It can be observed that the stable and unstable manifolds do not intersect. However, in Fig. 5(b), we consider the case $F = 0.081$, in which is shown that the invariant manifolds clearly intersect. The invariant manifolds for the simple Duffing oscillator are depicted in Fig. 5(c) with the forcing parameter $F = 0.075$. It is very interesting to note the similarity with the case in which a generalized nonlinear damping term is used, which is shown in Fig. 5(a). The equivalent system with $F = 0.081$ is shown in Fig. 5(d), with results very similar to Fig. 5(b). As a consequence, the computations agree with our theoretical discussions.

From Fig. 5, it may be observed that the invariant manifolds of both nonlinear oscillators are almost identical. This fact has dynamical

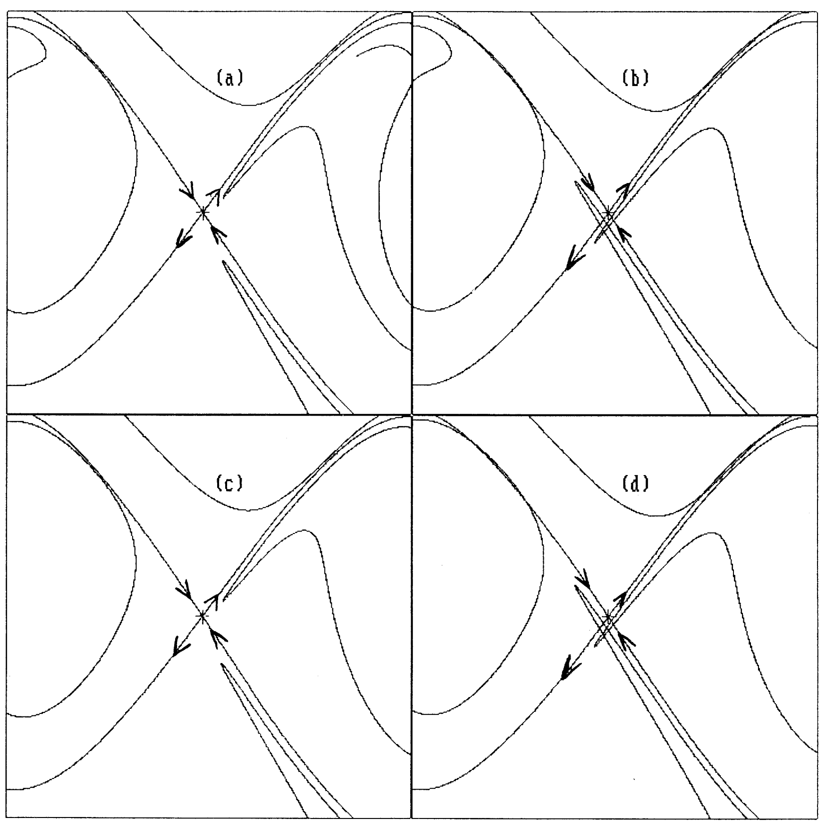


Fig. 5. Stable and unstable manifolds for different values of the parameters corresponding to the nonlinearly damped Duffing oscillator $\ddot{x} + \alpha_1\dot{x} + \alpha_3\dot{x}^3 - x + x^3 = F\cos\omega t$. (a) $\alpha_1 = 0.1$, $\alpha_3 = 0.01$, $F = 0.075$. (b) $\alpha_1 = 0.1$, $\alpha_3 = 0.01$, $F = 0.081$. (c) $\alpha_1 = 0.1034$, $\alpha_3 = 0$, $F = 0.075$. (d) $\alpha_1 = 0.1034$, $\alpha_3 = 0$, $F = 0.081$.

consequences, not only because we have detected the critical parameters for which the tangencies occur, but also because the basins of attraction of both systems are quite similar. In relation to these ideas, we have investigated the energy dissipation and the force- and frequency-response behavior for both oscillators near the critical parameters, showing that both of them have an almost indistinguishable free-decay behavior and identical steady-state forced-oscillation behavior.

4.2. *Pendulum*

A similar analysis can be done for the pendulum. When more nonlinear damping terms are introduced the pendulum equation becomes

$$\ddot{\theta} + \sum_{i=1}^{N} \alpha_{p_i} \dot{\theta} |\dot{\theta}|^{p_i - 1} + \sin\theta = F \cos \omega t\,, \qquad (29)$$

which contains a generalized nonlinear damping term. The Melnikov function results in

$$M^{\pm}(t_0) = -\sum_{i=1}^{N} 2^{p_i+1} \alpha_{p_i} B\left(\frac{1}{2}, \frac{p_i+1}{2}\right) + 2\pi F \,\mathrm{sech}\left(\frac{\pi\omega}{2}\right) \cos \omega t_0\,, \qquad (30)$$

and the critical parameter of the external perturbation is given by

$$F_{\mathrm{cp}} = \cosh\left(\frac{\pi\omega}{2}\right) \sum_{i=1}^{N} \frac{2^{p_i}}{\pi} \alpha_{p_i} B\left(\frac{1}{2}, \frac{p_i+1}{2}\right). \qquad (31)$$

The generalized nonlinearly damped pendulum is Melnikov-equivalent to the simple damped pendulum with equation

$$\ddot{\theta} + \mu\dot{\theta} + \sin\theta = F \cos \omega t\,, \qquad (32)$$

if the amplitude F and frequency ω of the external forcing are equal in both systems, and the parameter μ in (32) is a linear function of the set of

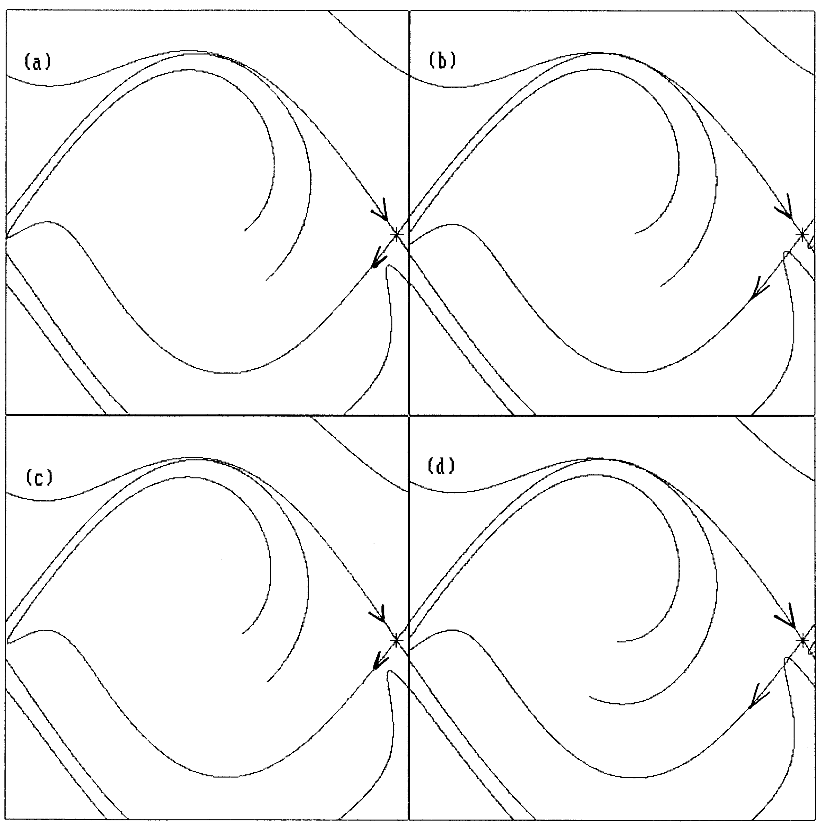


Fig. 6. Stable and unstable manifolds for different values of the parameters corresponding to the nonlinearly damped simple pendulum $\ddot{\theta} + \alpha_1\dot{\theta} + \alpha_3\dot{\theta}^3 + \sin\theta = F \cos \omega t$. (a) $\alpha_1 = 0.1$, $\alpha_3 = 0.01$, $F = 0.39$. (b) $\alpha_1 = 0.1$, $\alpha_3 = 0.01$, $F = 0.41$. (c) $\alpha_1 = 0.1267$, $\alpha_3 = 0$, $F = 0.39$. (d) $\alpha_1 = 0.1267$, $\alpha_3 = 0$, $F = 0.41$.

parameters α_{p_i} given by

$$\mu = \sum_{i=1}^{N} 2^{p_i-2} B\left(\frac{1}{2}, \frac{p_i+1}{2}\right) \alpha_{p_i} . \qquad (33)$$

4.2.1. *Numerical simulation*

The particular case of the generalized damped pendulum (29) studied here is

$$\ddot{\theta} + \alpha_1\dot{\theta} + \alpha_3\dot{\theta}^3 + \sin\theta = F\cos\omega t , \qquad (34)$$

which is Melnikov-equivalent to the simple system (32) if

$$\mu = \alpha_1 + \frac{8}{3}\alpha_3 . \qquad (35)$$

The critical parameter for the generalized nonlinearly damped pendulum (34), and its associated Melnikov-equivalent simple system, is

$$F_{cp} = \frac{4}{\pi}\cosh\left(\frac{\pi\omega}{2}\right)\left(\alpha_1 + \frac{8}{3}\alpha_3\right) . \qquad (36)$$

For the numerical computations, the parameters were taken to be $\omega = 1$, $\alpha_1 = 0.1$, $\alpha_3 = 0.01$ in the generalized system (34). The equivalent simple damped system has a damping coefficient $\mu = 0.12667$ according to the relation (35), and the critical parameter is $F_{\text{cp}} = 0.40468$.

As in the case of the Duffing oscillator, Fig. 6 shows the stable and unstable manifolds of the generalized system with $\omega = 1$, $\alpha_1 = 0.1$, $\alpha_3 = 0.01$, and the equivalent simple system with $\omega = 1$, $\mu = 0.12667$. In Fig. 6(a) appears the plot of the invariant manifolds when $F = 0.39$ for the generalized system, which is below the critical parameter. However in Fig. 6(b) the case $F = 0.41$, which is above the critical forcing, is shown. The invariant manifolds for the simple damped pendulum are shown in Fig. 6(c), which corresponds to the equivalent simple system with forcing parameter $F = 0.39$, that is, below the critical forcing. The same situation is described in Fig. 6(d) for the parameter value $F = 0.41$, which is above the critical forcing. Again, theory and computations agree.

Analogously as was described previously for the Duffing oscillator, we have observed the dynamical consequences of the similarities of the invariant manifolds for our damped pendulum equations. The results of our investigation show that the free-decay behavior and the steady-state forced-oscillation behavior are equivalent for both, the linearly damped and the nonlinearly damped pendulum.

5. Concluding Remarks

As it was pointed out in the introduction, dissipation plays an important role in dynamical systems. Indeed most dynamical systems may be classified as dissipative systems. In spite of the enormous efforts to analyze these systems, very few works appear to exist, in which nonlinear dissipation is considered. As a matter of fact, it seems that the effect of nonlinear dissipation on certain nonlinear dynamical systems has been missed or underestimated. In this work we analyze the Duffing oscillator and the simple pendulum, with nonlinear damping terms proportional to the power of the velocity. We have analytically computed, using the Melnikov method, the threshold parameters for which homoclinic or heteroclinic chaos is expected. Our analysis has provided some useful general expressions which can be of application to many different models in sciences and technology, where nonlinear damping terms proportional to the power of the velocity are included. For the nonlinearly damped Duffing oscillator, with a single nonlinear damping term proportional to the power of the velocity, our analytical results show that the critical parameter for which homoclinic chaos exist, decreases when the damping exponent, that is the power of the velocity, increases. For the nonlinearly damped simple pendulum with a single nonlinear damping term proportional to the power of the velocity, the analytical results show that the critical parameter increases when the damping exponent increases. Using the concept of Melnikov equivalence and the generalized nonlinear damping term, which includes several nonlinear damping terms, we have shown that the effect of nonlinear dissipation can be equivalent to a linearly damped nonlinear oscillator with a modified damping coefficient. The investigation of the energy dissipation and the force- and frequency-response behavior of the nonlinearly damped and the Melnikov equivalent linearly damped oscillators show that both of them have an almost indistinguishable free-decay and steady-state forced-oscillation behavior. Accordingly, all these results show the importance of studying the role of the nonlinear dissipative forces in dynamical systems.

Acknowledgments

The figures showing the stable and unstable manifolds of the nonlinear oscillators were computed with the software DYNAMICS [Nusse & Yorke, 1998]. This work was supported by the Ministry of Education and Culture, Spain, under project DGES PB96-0123.

References

Abramowitz, A. & Stegun, I. [1970] *Handbook of Mathematical Functions* (Dover, NY).

Awrejcewicz, J. & Holicke, M. M. [1999] "Melnikov's method and stick-slip chaotic oscillations in very weakly forced mechanical systems," *Int. J. Bifurcation and Chaos* **9**, 505–518.

Bikdash, M., Balachandran, B. & Nayfeh, A. [1994] "Melnikov analysis for a ship with a general roll-damping model," *Nonlin. Dyn.* **6**, 101–124.

De Mestre, N. [1990] *The Mathematics of Projectiles in Sport* (Cambridge University Press, Cambridge).

Falzarano, J. M., Shaw, S. W. & Troesch, A. W. [1992] "Application of global methods for analyzing dynamical systems to ship rolling motion and capsizing," *Int. J. Bifurcation and Chaos* **2**, 101–115.

Huilgol, R. R., Christie, J. R. & Panizza, M. P. [1995] "The motion of a mass hanging from an overhead crane," *Chaos Solit. Fract.* **5**, 1619–1631.

Koch, B. P. & Leven, R. W. [1985] "Subharmonic and homoclinic bifurcations in a parametrically forced pendulum," *Physica* **D16**, p. 1.

Lawden, D. [1989] *Elliptic Functions and Applications* (Springer-Verlag, Berlin).

Lim, Y. F. & Chen, K. [1998] "Dynamics of dry friction: A numerical investigation," http://xxx.lanl.gov/list/cond-mat/9803217.

Linz, S. J. [1995] "A simple nonlinear system: The oscillator with quadratic friction," *Eur. J. Phys.* **16**, 67–72.

Litak, G., Spuz-Szpos, G., Szabelski, K. & Warminski, J. [1999] "Vibration of externally-forced Froude pendulum," *Int. J. Bifurcation and Chaos* **9**, 561–570.

Moon, F. C. & Li, G.-X. [1985] "Fractal basin boundaries and homoclinic orbits for periodic motions in a two-well potential," *Phys. Rev. Lett.* **55**, 1439–1442.

Nayfeh, A. H. & Balachandran, B. [1995] *Applied Nonlinear Dynamics. Analytical, Computational and Experimental Methods* (Wiley, NY).

Nusse, H. E. & Yorke, J. A. [1998] *Dynamics: Numerical Explorations* 2nd edition (Springer, NY).

Pfenniger, D. & Norman, C. [1990] "Dissipation in barred galaxies: The growth of bulges and central mass concentrations," *The Astrophys. J.* **363**, 391–410.

Ravindra, B. & Mallik, A. K. [1994a] "Stability analysis of a non-linearly damped Duffing oscillator," *J. Sound Vibr.* **171**, 708–716.

Ravindra, B. & Mallik, A. K. [1994b] "Role of nonlinear dissipation in soft Duffing oscillators," *Phys. Rev.* **E49**, 4950–4954.

Sanjuán, M. A. F. [1996] "Remarks on transitions order-chaos induced by the shape of the periodic excitation in a parametric pendulum," *Chaos Solit. Fract.* **7**, 435–440.

Sanjuán, M. A. F. [1999] "The effect of nonlinear damping on the universal escape oscillator," *Int. J. Bifurcation and Chaos* **9**, 735–744.

Wiggins, S. [1990] *Introduction to Applied Nonlinear Dynamical Systems and Chaos* (Springer-Verlag, Berlin).

Xie, F. & Hue, G. [1995] "Periodic solutions and bifurcation behavior in the parametrically damped two-well Duffing equation," *Phys. Rev.* **E51**, 2773–2778.

Appendix A

All the integrals used for the computation of Melnikov functions are:

$$\int_{-\infty}^{+\infty} dt\, \frac{\sinh^{\mu}(t)}{\cosh^{\nu}(t)} = B\left(\frac{\mu+1}{2}, \frac{\nu-\mu}{2}\right) \tag{A.1}$$

$$\int_{-\infty}^{+\infty} dt\, \mathrm{sech}\, At \cos Bt = \frac{\pi}{A}\, \mathrm{sech}\left(\frac{\pi B}{2A}\right) \tag{A.2}$$

$$\int_{-\infty}^{+\infty} dt\, \mathrm{sech}\, At \tanh At \sin Bt = \frac{\pi B}{A^2}\, \mathrm{sech}\left(\frac{\pi B}{2A}\right) \tag{A.3}$$

The Euler Beta functions are related to the Gamma Euler Function through the expression:

$$B(r, s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}\,. \tag{A.4}$$

Remember that the Euler Gamma function has the following property: $\Gamma(n+1) = n\Gamma(n)$, where $\Gamma(n+1) = n!$, $0! = 1$, if $n = 0, 1, 2, \ldots$, and $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(n+1/2) = (1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)/2^n)\sqrt{\pi}$.