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WADA BASINS AND UNPREDICTABILITY IN HAMILTONIAN AND DISSIPATIVE SYSTEMS

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Prediction is one of the fundamental goals of science. When prediction is lost, it can be thought that one of the foundations of science is shattered. The notion of a chaotic system and the sensitive dependence on initial conditions implies a certain lack of prediction on the time evolution along an orbit. However we do not speak here about this temporal prediction, but about an extreme dependence on the initial conditions that fractal structures in phase space impose, and that obstructs the prediction of the final state of the system.

1. Introduction

> More individuals are born than can possibly survive. A grain in the balance will determine which individual shall live and which shall die, which variety or species shall increase in number, and which shall decrease, or finally become extinct.
> — Darwin, *The origin of species*, c. 14, 1859

The phenomenon of chaotic scattering is usually associated to the dynamics of open Hamiltonian systems which possess chaotic properties. What usually happens is that a particle moves to and fro during certain time in a bounded region commonly called **scattering region**, and eventually escapes towards infinity through one of the existing exits. Two-dimensional Hamiltonian systems have been studied by many researchers, since they are widely used in modeling different physical phenomena. Some applications are the analysis of the escape of stars in galaxies,[1] the dynamics of ions in electromagnetic traps,[2] or the interaction between the magnetic tail of the Earth and the solar wind,[3] to cite just a few. All these applications are manifestations of **chaotic scattering**, which mainly consists of the interaction of a particle with a system which scatters it, in such a way that the final conditions of velocity and direction possess an extreme dependence on the initial conditions, which is a sign of chaotic behavior.

Usually there is a threshold value of the energy, which is called *escape energy*. Below it, the orbits are bounded and the particles located in the scattering region cannot escape. However, when the energy is above this threshold value, several exits appear and the particles are able to escape towards infinity through any of them.

When we consider a Hamiltonian system, the total energy is conserved and hence we cannot speak about attractors, neither basins of attraction. A **basin of attraction** is defined as the set of initial conditions which are attracted toward a certain **attractor**, and they only exist in **dissipative systems**. When two attractors coexist in a certain region of phase space, we have two basins, which are separated by a **basin boundary**. This basin boundary could be a curve, but it can also be a **fractal**. While we cannot speak about attractors in Hamiltonian systems, we can speak about escape or exit basins in an analogous manner to what basins of attraction are for dissipative systems. An **exit basin** is the set of initial conditions leading to a certain exit.

These basins might not be only fractal, as it is possible for them to possess the stronger property of Wada.[5,6] We say that a basin verifies the **Wada property**, which might hold when there exist at least three basins, when any of its boundary points belongs simultaneously to the boundary of the other two basins. Thus, if a dynamical system verifies the Wada property, the unpredictability is even larger than when there are only fractal basin boundaries. If a trajectory starts very close to a boundary point, it will not be possible to predict beforehand its future behavior, since its initial conditions could belong to any of the three basins.

The first example of a system with this property was reported by the Japanese mathematician Yoneyama in 1917,[7] who attributed the idea to a certain Mr. Wada. Yoneyama took the name from this person, and used it to designate what is known as the "Wada lakes", which is a rather useful example of how to build three regions holding this property. Logically, the boundaries of these sets verify very unusual topological properties.

A beautiful experiment showing the Wada property was reported recently in an optical system possessing chaotic scattering published in Nature.[8]

2. Wada Basins in the Hénon-Heiles System

In particular, we study the exit basins of the **Hénon-Heiles Hamiltonian**, which is well known as a model of an axy-symmetric galaxy.[4] The Hénon-Heiles system is given by the equation

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3, \tag{1}$$

and constitutes a paradigm for Nonlinear Dynamics of Hamiltonians systems. It is a two-dimensional time independent dynamical system, it possesses three different exits for orbits above the escape energy, and it has a $2\pi/3$ symmetry. Moreover, the associated dynamics is quite unpredictable, since the boundary separating their exit basins is not a smooth curve.

We have found in Ref. 9 that the Hénon-Heiles system possesses Wada basins. Furthermore, we believe that the Wada property is a general feature for open two-

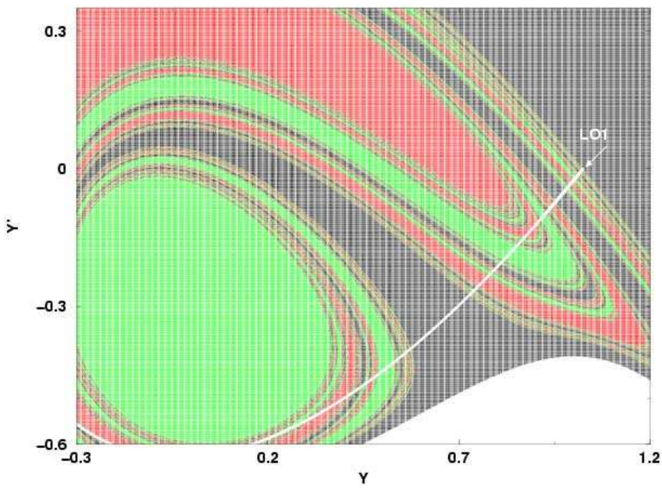

Fig. 1. The unstable manifold of the only accessible unstable periodic orbit crosses all the basins in this zoom of the exit basin diagram of the system. Therefore, the Hénon-Heiles Hamiltonian verifies the property of Wada.[9]

dimensional Hamiltonian systems with three or more escapes. In fact, this kind of models is widely used for modeling many astrophysical problems, although the underlying ideas are of application in many cases, where some kind of **transient chaos**[10] is present.

3. Wada Basins in the Duffing Oscillator

Another objective of this paper is to show that the **Duffing oscillator** presents the Wada property.[11] The Duffing oscillator is a well known model of a nonlinear oscillator, and it is applicable to modeling many systems in science and engineering. As a matter of fact, it is considered as a paradigm for Nonlinear Dynamics of dissipative systems.

It can be understood as a model for the one-dimensional motion of a unit mass particle inside a symmetrical double-well potential, with dissipation and an external periodical forcing. The equation that we have used is

$$\ddot{x} + \delta\dot{x} - \alpha x + \beta x^3 = \gamma \cos \omega t \,. \tag{2}$$

The variable $x(t)$ represents the position in time t, being δ the damping coefficient. The parameters γ and ω represent the amplitude and the external perturbation forcing. We use the parameters $\delta = 0.15$, $\alpha = \beta = \omega = 1$ and vary γ, and we have concentrated in the amplitude range $0.24 \leq \gamma \leq 0.26$, where **three attractors coexist** in phase space.

As it was formerly commented, one of the main consequences of the fact that these systems possess Wada basins is the intrinsic difficulty to make predictions, in such a way that we could not know beforehand to which attractor the systems

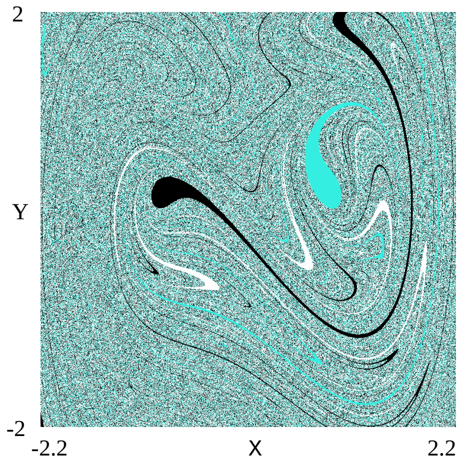

Fig. 2. The picture shows the basin of attraction diagram. A fine grid of 960×960 of initial points is considered and different colors are chosen according to which attractor an initial condition goes to.

go for a given initial condition. This has an enormous importance, since we are used to the idea of classical determinism, where once an initial condition is fixed, automatically we know the evolution of the orbit. From an experimental point of view, to fix an initial condition with arbitrarily high precision is not possible, and consequently a serious problem for the prediction of physical systems is derived.

These ideas suppose in fact a challenge to the classical ideas of determinism. An interesting discussion around the physical consequences of certain fractal basin boundaries and **unpredictability** appears in Ref. 12.

4. Applications in Physics and Other Applied Sciences

As it was mentioned earlier, the Wada property might hold for Hamiltonian systems as for dissipative systems, with similar consequences concerning the unpredictability of the final state of the system. There are results confirming that the property also holds for the forced pendulum,[5] in a model of three billiards,[13] in problems of chaotic advection of a fluid flow[14,15] and in ecological models.[16]

There are very interesting applications in relation to multispecies competition, which is a very active research field in ecology and biodiversity.[17] In this field, a strong unpredictability about the survival species is present. This, perhaps, explains the quotation from Darwin.

5. Other Problems

As a natural continuation to the work done for Hamiltonian systems, we have also explored another novel type of basins, which we call **uncertain basins**. We believe that when the size of the exits diminishes tending to zero, then the unpredictability increases a lot, in such a way that the information about the future of the system is lost. This is the first time that such a phenomenon is described for Hamiltonian systems.[18]

6. Conclusions

Two paradigmatic dynamical systems in Nonlinear Dynamics are studied, one of them a Hamiltonian system, the Hénon-Heiles system, and the other one dissipative, the Duffing oscillator. We show that both possess Wada basins, affecting the unpredictability of the final state of the system, in such a way that in order to predict its final state, in some cases only the probabilistic approach is possible.

Moreover, several open problems show the interest of exploring these fractal structures in other dynamical systems, in order to clarify their physical consequences.

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References

1. G. Contopoulos, H. E. Kandrup and D. Kaufman, *Physica* **D64**, 310 (1993).
2. G. Z. K. Horvath, J. L. Hernández Pozos, K. Dholakia, J. Rink, D. M. Segal and R. C. Thompson, *Phys. Rev.* **A57**, 1944 (1998).
3. J. Chen, J. L. Rexford and Y. C. Lee, *Geophys. Res. Lett.* **17**, 1049 (1990).
4. M. Hénon and C. Heiles, *Astron. J.* **69**, 73 (1964).
5. J. Kennedy and J. A. Yorke, *Physica* **D51**, 213 (1991).
6. H. E. Nusse and J. A. Yorke, *Science* **271**, 1376 (1996).
7. K. Yoneyama, *Tohoku Math. J.* **11–12**, 43 (1917).
8. D. Sweet, E. Ott and J. A. Yorke, *Nature* **399**, 315 (1999).
9. J. Aguirre, J. C. Vallejo and M. A. F. Sanjuán, *Phys. Rev.* **E64**, 066208 (2001).
10. T. Tél, in *Experimental Study and Characterization of Chaos*, ed. Hao Bailin (World Scientific, Singapore, 1990), p. 149.
11. J. Aguirre and M. A. F. Sanjuán, *Physica* **D171**, 41 (2002).
12. J. Sommerer, *Johns Hopkins APL Tech. Dig.* **16(4)**, 333 (1995).
13. L. Poon, J. Campos, E. Ott and C. Grebogi, *Int. J. Bifurcation Chaos* **6**, 251 (1996).
14. Z. Toroczkai, G. Karoly, A. Pentek, T. Tel, C. Grebogi and J. A. Yorke, *Physica* **A239**, 235 (1997).
15. A. Witt, R. Braun, F. Feudel, C. Grebogi and J. Kurths, *Phys. Rev.* **E59**, 1605 (1999).
16. J. Vandermeer, L. Stone and B. Blasius, *Chaos, Solitons and Fractals* **12**, 265 (2001).
17. J. Huisman and F. J. Weissing, *The American Naturalist* **157**, 488 (2001).
18. J. Aguirre and M. A. F. Sanjuán, Preprint-2002.