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Controlling chaos in a fluid flow past a movable cylinder

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Abstract

The model of a two-dimensional fluid flow past a cylinder is a relatively simple problem with a strong impact in many applied fields, such as aerodynamics or chemical sciences, although most of the involved physical mechanisms are not yet well known. This paper analyzes the fluid flow past a cylinder in a laminar regime with Reynolds number, *Re*, around 200, where two vortices appear behind the cylinder, by using an appropriate time-dependent stream function and applying non-linear dynamics techniques. The goal of the paper is to analyze under which circumstances the chaoticity in the wake of the cylinder might be modified, or even suppressed. And this has been achieved with the help of some indicators of the complexity of the trajectories for the cases of a rotating cylinder and an oscillating cylinder.

1. Introduction

Despite the fact that the fluid flow past a cylinder has been studied for well over one hundred years, and that the geometrical configuration is a particularly simple one, this problem is still under intensive research today [1]. Much attention has been paid to this problem also because it might be considered a paradigm of pattern formation and transient chaos. The principal motivation for the study of this problem here lies at the question of under which circumstances the transient chaos at the wake of the cylinder might be reduced or even suppressed. Therefore the main objectives of the present paper are to investigate how a rotation or a transversal vibration affects the flow of an incompressible, viscid, time-dependent fluid flow past a cylinder in the laminar vortex shedding regime and to understand the mechanisms of the vortex shedding suppression. The Lagrangian dynamics provides a good approach to study the structure of the flow. Unlike the Eulerian description, which characterizes the velocity field, the Lagrangian one emphasizes the motion of the individual fluid particles by following them along the pathlines.

This means that using a proper stream function, ψ, a Hamiltonian approach can be used and, assuming incompressibility, the equations of motion of the velocity field for a passive advected dye particle in the x–y plane have the form:

$$\begin{aligned} u(x,y,t) &= \frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{\partial\psi(x,y,t)}{\partial y} \\ v(x,y,t) &= \frac{\mathrm{d}y(t)}{\mathrm{d}t} = -\frac{\partial\psi(x,y,t)}{\partial x} \end{aligned} \tag{1}$$

For a steady flow the pathlines of the individual fluid particles coincide with the streamlines of the flow. However this is not so for a time-dependent stream function. Even for simple two-dimensional flows, such as time periodic flows, the

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trajectories can be very complicated. This phenomenon has been termed *chaotic advection* or *Lagrangian turbulence*, and it can be analyzed using the appropriate techniques from Hamiltonian non-linear dynamics. In fact what appears at the wake of the cylinder, when the two vortices are acting periodically, is a sort of transient chaos from the non-linear dynamics point of view, since the flow is an open flow.

Hence, in this paper we analyze a fluid flow past a cylinder in a laminar regime with Reynolds number, *Re*, around 200. And in particular, our work has been focused in the analysis of how the chaoticity in the wake of the cylinder is modified or even suppressed when a rotation or a transversal vibration on the cylinder is considered.

The basic idea of flow control is to look for situations where there are sharp gradients or rapid variations. In such scenarios, properly applied small perturbations can lead to huge changes and, therefore, opportunities for controlling the flow arise. A typical region is the shear layer, the region where, due to the viscous effects, there is a velocity gradient (or profile) perpendicular to the streamlines, and, depending on this profile, an instability can be found. Additional regions where flow control can be carried out are vortices, which contain steep gradients of velocity and pressure, and areas of interaction of vortices with physical surfaces.

Several relevant problems are affected by such instabilities. Fluctuating forces may enhance mixing in chemical processes or cause structural vibrations, triggering structure failures. In fact, these oscillations due to the shedding turbulent vortices induced in a periodic manner has been reported as one plausible reason to the Tacoma narrows bridge collapse, as stated for instance in [2].

In order to understand under which circumstances the complexity of the trajectories of passive tracers in a fluid flow past a cylinder is reduced or even suppressed, we analyze two control mechanisms. The first one consists of moving the cylinder in a rotational way, whereas the second one consists of moving it in a direction perpendicular to the incoming flow. A physical experiment concerning these ideas and showing the possibility of reducing the velocity fluctuations in a Kármán vortex street by introducing mechanical vibrations with the right amplitude and phase is described by Wehrmann in [3].

The remaining of the paper is organized as follows. In Section 2, we briefly describe the well-studied case of a fluid flow past a static cylinder by using an appropriate stream function, and where the hamiltonian dynamics techniques are used. In Section 3, we will describe how a clockwise rotation of the cylinder might modify the chaotic region in the wake of the cylinder. Analogous results can be obtained by making the cylinder oscillate in a single direction, as described in Section 4. Finally, we present a discussion of our results.

2. Fluid flow past a static cylinder

When a circular cylinder is placed with its axis perpendicular to the flow direction, depending on the Reynolds number, it is possible to observe a periodic shedding of vortices in the wake of the cylinder (i.e., the turbulent area behind the cylinder). This area is where the viscosity is truly important.

At low *Re* ($Re < 47$) the wake behind a static cylinder comprises a steady recirculation region with two vortices symmetrically attached to the cylinder, whose size grows when *Re* is increased.

When $Re < 200$, vortex shedding occurs in the near wake behind the cylinder accompanying a periodically oscillating lift force. Each half of the period, a vortex breaks off, so the surface pressure distribution around the cylinder changes dramatically as the cylinder experiences a sudden impulse. These vortices generate alternating high and low pressure regions on the lee side of the body in such a way that the cylinder experiences a periodic net force. This oscillation may amplify the vortex shedding, because now the cylinder itself is moving in the flow, forcing the vortex street to occur with large amplitude. The frequency of the vortices is given by the so called Strouhal number S_t, and the vortices produced in this manner are termed Strouhal vortices.

Finally for higher values of *Re*, that is, when $Re > 200$, the flow becomes three-dimensional and turbulent.

We are going to focus on the two-dimensional shedding regime of $Re \approx 200$, where the experimental relationship between Strouhal and Reynolds numbers is quite well determined ($S_t \approx 0.2$) and the two-dimensional model is still valid. In our study we have used the advective approach described in [5]. It is known that, when solving numerically the Navier–Stokes equations for the flow past a cylinder, one must take into account that there is a difficulty with the setup of the boundary conditions due to the special configuration of the problem. Depending on the treatment at large distance of the cylinder, the value of many physical properties such as lift and drag coefficients can vary. However, by applying Eq. (1) we can get accurate numerical results, taking advantage of the fact that the same results can be easily obtained by letting a test particle evolve in the system. A major point is to realize that this approach actually leads to the same results as the Navier–Stokes equations. In a strict sense, the stream function should be derived from the

Navier–Stokes equations and, indeed, it can be numerically calculated from the Navier–Stokes velocity field. However, by using a proper analytical stream function that fits the properties of the real solution, it is expected that the obtained advective behavior be close enough to the real one. The main point is, therefore, to model the real conditions of the problem with a suitable ad hoc stream function, in order to simulate the cylinder, the incoming flow and the asymmetric periodic break off of the vortices.

As a starting point we will take the analytical model of the stream function described in [5,9]. This function simulates the fluid flow past a static cylinder in the conditions described above. In this section, we briefly describe and explain the terms and components of this model stream function, which will be modified in the next sections. The model stream function for the static case can be written as

$$\psi(x,y,t) = f(x,y)g(x,y,t). \tag{2}$$

The first factor

$$f(x,y) = 1 - \exp(-a((x^2+y^2)^{1/2}-1)^2), \tag{3}$$

yields to the correct non-slip boundary condition for the overall function at the cylinder surface. The cylinder radius, R, has been taken to be unity and the coefficient $a^{-1/2}$ plays the role of the width of the boundary layer. This form ensures that the tangential velocity linearly tends to zero as expected in the boundary layer and the radial component of the velocity vanishes quadratically.

The second factor, $g(x,y,t)$, contains the contributions of the vortices and of the background flow with velocity u_0 and can be written as

$$g(x,y,t) = -wh_1(t)g_1(x,y,t) + wh_2(t)g_2(x,y,t) + u_0 ys(x,y). \tag{4}$$

As it can be seen, the first two terms describe the alternating birth, evolution and damping of two vortices of equal strength but opposite sign. The quantities w and $h_i(t)$ stand for the overall vortex strength and amplitudes, respectively. Because of the alternating character, one has $h_2(t) = h_1(t - T_c/2)$ where T_c denotes the time period of the flow and $h_1(t) = |\sin(\pi t/T_c)|$. In this model the vortex centers are assumed to move parallel to the x-axis and with a constant velocity. Their x-coordinates, without the influence of the boundary condition prefactor $f(x,y)$, are expected to evolve with time as

$$\begin{aligned} x_1(t) &= 1 + L[(t/T_c)\mathrm{mod}1], \\ x_2(t) &= x_1(t - T_c/2), \end{aligned} \tag{5}$$

and the y-coordinates are constants,

$$y_1(t) = -y_2(t) = y_v. \tag{6}$$

Both vortices move a distance L during time T_c and then die out. Thus, vortex 1 starts at ($x = 1$, $y = y_v$) with zero amplitude at $t = 0$, and at this time, vortex 2 is just in its most developed state at ($x = 1 + L/2$, $y = y_v$). The contribution of the vortices to the stream function is represented by the equation

$$g_i(x,y,t) = \exp(-R_0[(x - x_i(t))^2 + \alpha^2(y - y_i(t))^2]), \tag{7}$$

where $R_0^{-1/2}$ is the characteristic linear size of the vortices and α is the elongation factor. The last term in Eq. (4) gives the contribution to the stream function from the background flow. The factor

$$s(x,y) = 1 - \exp(-(x-1)^2/\alpha^2 - y^2), \tag{8}$$

simulates in a phenomenological manner the shielding of the background flow just behind the cylinder.

The values of the parameters used throughout this contribution are $\alpha = 2$, $R_0 = 0.35$, $L = 2$, $a = 1$ and $y_v = 0.3$. Since we want to set the time unit equal to the system period, that is $T_c = 1$, and the vortices should be around 7 times slower than the incoming flow, the velocity of the incoming flow must be 14.

The complicated motion of each flow particle in the flow is organized around the invariant chaotic saddle formed by all the periodic orbits lying in the wake and their heteroclinic and homoclinic connections. It is known that the low-order periodic orbits play a fundamental role in the system. The long-lived scattering trajectories come close to periodic orbits, and any periodic orbit can be built up from segments of the low-order ones. However, the shorter periodic orbits generated by the vortices are not sufficient for explaining the shadowing of the complicated orbits by the low-order ones. In addition, the wall of the obstacle consisting of a continuum of parabolic points has to be included as a further low-order periodic orbit. Therefore, the invariant set has two components. The first one contains the short hyperbolic orbits as well as the ones shadowed by them. The second one contains the obstacle wall as well as the periodic orbits shadowed by it. See [5] for details. In the following sections, we describe how such structures are modified by rotating the cylinder or by making it oscillate in a direction perpendicular to the incoming flow.

3. Fluid flow past a rotating cylinder

The rotation of a cylinder in a viscous flow is expected to modify the wake flow pattern and vortex shedding configuration. It may reduce the flow-induced oscillation or increase the lift force. This latter is termed Magnus force and it is due to the asymmetric displacement of the boundary layer caused by the combined spinning and flow past the cylinder. The cylinder can impart a spinning motion only to a very thin layer next to the surface. This motion affects the way in which the flow separates from the surface in the space behind the cylinder. Boundary layer separation is delayed on the side of the spinning object that is moving in the same direction as the free stream flow, while the separation occurs prematurely on the side moving against the free stream flow. Then, the wake shifts towards the side moving against the free stream flow, so the flow is deflected and the resulting change in momentum causes a force in the opposite direction. This net force is referred to as lift. For clockwise rotation, the stagnation points are moved downwards

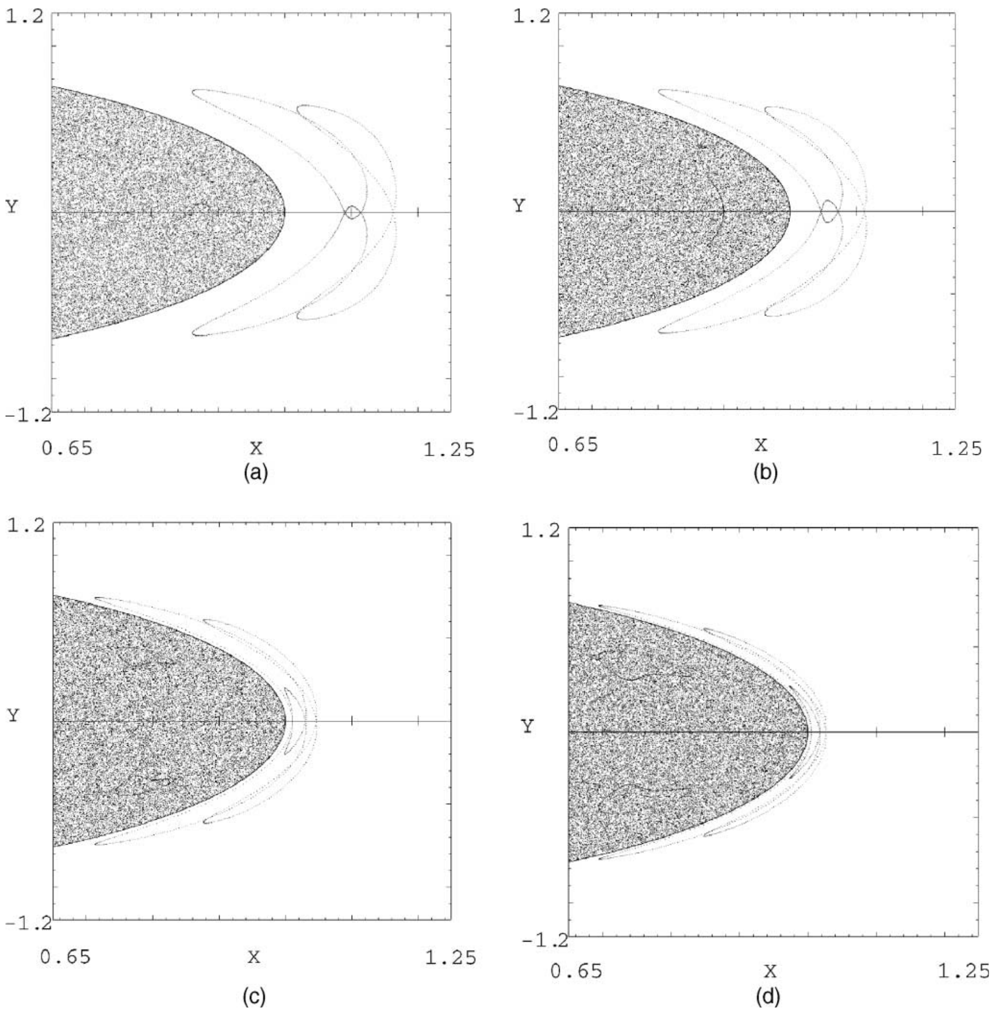

Fig. 1. The shape of the one-periodic orbits in the wake of the cylinder as the rotational parameter is increased. The different values of ω are (a) upper-left panel, 0.00, (b) upper-right panel, 0.10, (c) lower-left panel, 0.25 and (d) lower-right panel, 0.30.

and remain at opposite locations on the circle. If the rotation is counterclockwise, the net force is called drag and the stagnation points are moved upwards.

Some results from solving directly the Navier–Stokes equations are found in the literature [6,7]. In particular, in [7] a general description of the system can be found. However, only very recently (see [4]) experimental and numerical results have been shown to agree.

One of the goals of this contribution is the analysis of how the rotational motion affects the chaotic structures described in [9]. This requires two modifications in the stream function of Eq. (4). The first one is in the geometrical part, since now the non-slip condition leads to a velocity vector that must be defined in accordance with the cylinder clockwise rotation at speed ω. The second one is in the background flow term, as a general vorticity term must be added for generating the desired overall circulation and Magnus lift. The shielding factor is not modified. Since we are going to deal with only small ω values, the vortices are considered to be not affected by the overall circulation. Thus, their terms will also remain unaltered.

Functions $f(x,y)$ and $g(x,y,t)$ are modified as follows:

$$\begin{aligned} f(x,y) &= \omega((x^2+y^2)^{1/2}-1)+1-\exp(-a[(x^2+y^2)^{1/2}-1]^2), \\ g(x,y,t) &= -wh_1(t)g_1(x,y,t)+wh_2(t)g_2(x,y,t)+\left[u_0y+\frac{\omega}{2\pi}\ln\sqrt{x^2+y^2}\right]s(x,y). \end{aligned} \tag{9}$$

Several small values for ω are set in order to see how the structures described in [5,9] are modified. In the static case, the complicated motion of each flow particle is organized around the invariant chaotic saddle formed by all the periodic orbits lying in the wake and their heteroclinic and homoclinic connections. There was a hyperbolic effect in short time scales due to an infinity of strictly unstable periodic orbits, and another non-hyperbolic effect that affects the long time behavior of the tracers due to the obstacle wall.

We have computed the 1-periodic orbits for several small rotation parameters, as they are the building blocks for more complicated behavior. As it can be seen in Fig. 1, we found two different orbit types: a pair of symmetric lobes and one 8-shaped orbit. The eigenvalues associated to both types are modified as the rotation is increased, as is illustrated in Table 1 for the lobes and in Table 2 for the 8-shaped orbit. When a rotation is applied to the cylinder, the periodic orbits are modified in such a way that they are stretched and, at the same time, moved towards the cylinder. Meanwhile, the first eigenvalues are decreased and even converted into complex ones. In addition, the rear stagnation point, that caused the very long delays in the static case, is moved downwards (Magnus effect).

We have also computed the time-delay plots as a valid indicator of the chaoticity of the orbits. These plots are computed by starting the tracer dye from the same initial condition (in our case, $x_0 = -5.0$ and $y_0 = 0.01$) for different starting times, t_0, from 0 to T_c. Thus, each tracer finds the system in a different state. By plotting the difference between the time the particle should reach a given abscissa ($x_f = 10$ in our case) with and without bluff body, termed as δt, versus t_0, we obtain the set of figures with fractal structure that can be seen in Fig. 2. The peaks are associated to longer intervals when the particle is trapped by a periodic orbit, and in the limit, an infinite peak means the particle is trapped

Table 1
Eigenvalues of the lobe shaped 1-periodic orbits for several rotational parameters ω

ω	Λ_1	Λ_2
0.00	−5.33	0.18
0.10	−4.34	−0.23
0.25	−0.44 +i0.90	−0.44 −i0.90
0.30	−0.34 +i0.94	−0.34 −i0.94

Table 2
Eigenvalues of the 8-shaped 1-periodic orbits for several rotational parameters ω

ω	Λ_1	Λ_2
0.00	23.83	0.04
0.10	19.42	0.05
0.25	6.29	0.16
0.30	3.24	0.3

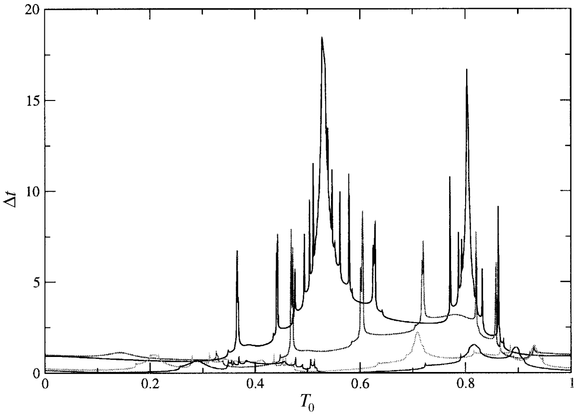

Fig. 2. Delay plots for the initial condition ($x_0 = -5.0$, $y_0 = 0.01$) at different starting times, t_0, as the rotational parameter is increased (the different values of ω are 0.00, 0.10, 0.25 and 0.30). Note that as ω is increased the height of the peaks are lowered. This shows that the trapping time of the trajectories decreases as ω increases and hence their chaoticity diminishes.

forever. It can be seen how the highest peaks disappear, as the rotation increases, so less chaoticity is found for the given initial condition, but a certain level of fractality still remains. In our model, the peaks are caused by the run near hyperbolic periodic orbits and by orbits that pass near the rear stagnation point. This point is slightly pushed downwards as ω increases. The general complexity of the figure is kept, but δt decreases as ω increases. As the flow is impelled by the general rotation, the particle moves faster and finds the vortices as if they were moving more slowly. In the limit, no transient chaos is expected since the particle moves so fast that the system is found like if it were on a time non-dependent potential (static vortices).

Finally, we have also traced some stable and unstable manifolds of the 1-periodic orbits. In Fig. 3 these sets associated to the prior periodic orbits are clearly observed and they have also been stretched around the cylinder surface, showing how even small rotations lead to a change in such sets. Consequently, for higher values of the rotation this set is strongly affected.

The effect of the rotation is to change substantially the streamline pattern near the surface of the cylinder. In the case of viscous flows, closed streamlines will always exist for all non-zero values of the rotational speed ω.

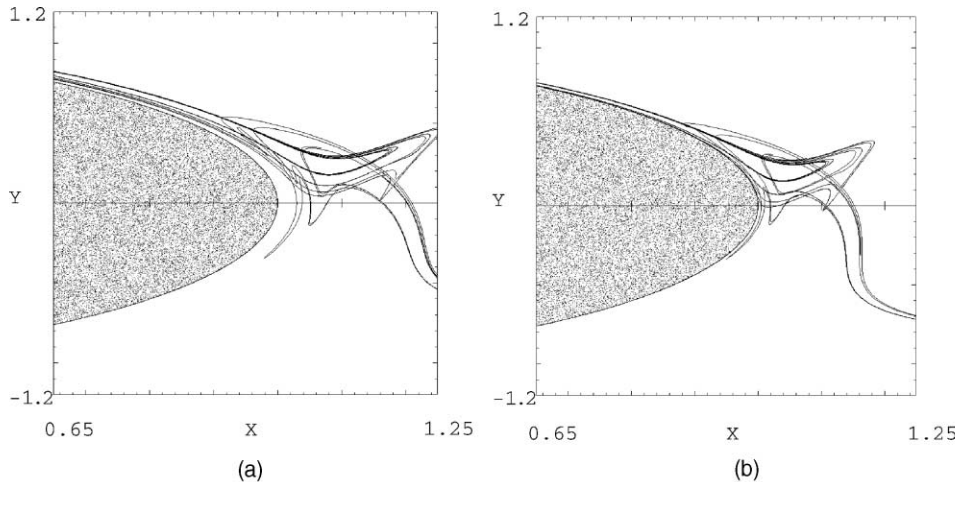

Fig. 3. Stable and unstable manifolds associated to the lobe-shaped one-periodic orbits of Fig. 1, for two different values of ω: (a) left panel, 0.00, (b) right panel, 0.10.

4. Fluid flow past an oscillating cylinder

In this section we are going to study the influence of the transversal oscillating motion of the cylinder placed perpendicularly to the incoming hydrodynamical flow described in the introduction. We want to simulate transversal harmonic oscillations of the cylinder of the type $A\sin(2\pi ft)$, where A and f represent the amplitude and the frequency of the oscillations, respectively.

In order to avoid problems related to the boundary condition at the cylinder's surface, we modify the factor $g(x,y,t)$ of the stream function given by Eq. (2). In particular, we set a new non-static frame origin at the center of the cylinder, so that the background flow oscillates in the vertical direction in an opposite way to the motion of the reference system (the cylinder itself), which is $A2\pi f\cos(2\pi ft)$. It is important to note that the inertial phenomena are neglected since the amplitude of the vertical vibrations of the background flow is small. This means that in a first approach the quantities that describe the overall vortex strength and amplitudes (see Eq. (4)) do not change. We assume that the vortices move with the cylinder in the original reference system. Thus, when we describe the model stream function from the new reference frame in motion, the vortices do not move. As a consequence, the quantities $g_i(x,y,t)$ do not change with respect to the ones of Eq. (7). The only change must be done in the background flow because in this case it has a uniform motion in the horizontal direction, but an oscillating motion in the vertical direction. Thus, to model this modification, we replace the term $u_0ys(x,y)$, in Eq. (4), by the new term $s(x,y)[u_0y + A2\pi fx\cos(2\pi ft)]$. Thus, the complete $g(x,y,t)$ term can be written as

$$g(x,y,t) = -wh_1(t)g_1(x,y,t) + wh_2(t)g_2(x,y,t) + s(x,y)[u_0y + A\omega x\cos(\omega t)]. \tag{10}$$

We have computed the time-delay plots of a particle placed at the initial condition (x_0, y_0) for different starting times, t_0. The initial point is chosen as in Section 3, that is, ($x_0 = -5.0$ and $y_0 = 0.01$). Fig. 4 represents the difference between the time the particle reaches the abscissa $x_f = 10$ with and without bluff body, that is, δt, versus the initial time t_0. As in the previous section, t_0 varies from 0 to T_c, being $T_c = 1$. Notice that δt has been evaluated for two different situations. In the first one (curve in black colour in Fig. 4) a static cylinder is considered, whereas in the second one (curve in gray colour in Fig. 4) the cylinder is moving harmonically with the same frequency as the frequency of the vortices and with an amplitude of 0.2 length units. One observation is how the chaoticity diminishes as the cylinder oscillates, for this initial condition. Notice also that the area below the curve is higher for the static situation. This is due to the choice of the initial point close enough to the y-coordinate corresponding to the center of the cylinder. Such trajectories, that experience most of the effect of the vortices in the static situation, become clearly less complex due to the cylinder motion. On the contrary, the complexity of the trajectories which are more distant of the y-coordinate corresponding to the center of the cylinder may increase. This phenomenon can be observed by comparing Fig. 5(a) and Fig. 5(b). In both plots δt is represented at different colours over a grid in which the horizontal axis represents the different initial times, t_0, at which the particles are placed at $x_0 = -5$, and the vertical axis represents the different values of y_0 that have been considered. Fig. 5(a) shows the case of the static cylinder, whereas Fig. 5(b) shows the case of the oscillating cylinder which moves with the same frequency as the frequency of the vortices and with an amplitude of 0.2 length units.

In summary, oscillating the cylinder vertically provides a means of reducing the complexity of a set of desired trajectories, at the expense of increasing the complexity of other trajectories.

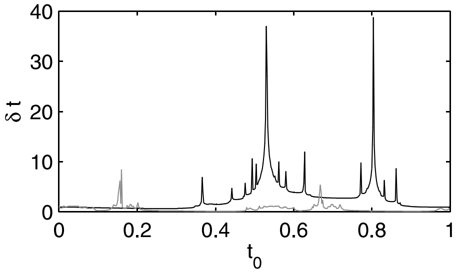

Fig. 4. Delay plots for the initial condition ($x_0 = -5.0$, $y_0 = 0.01$) at different starting times for the case of the static cylinder (curve in black) and for the case of the oscillating cylinder with amplitude equal to 0.2 length units and frequency the same as the one of the vortices (curve in gray).

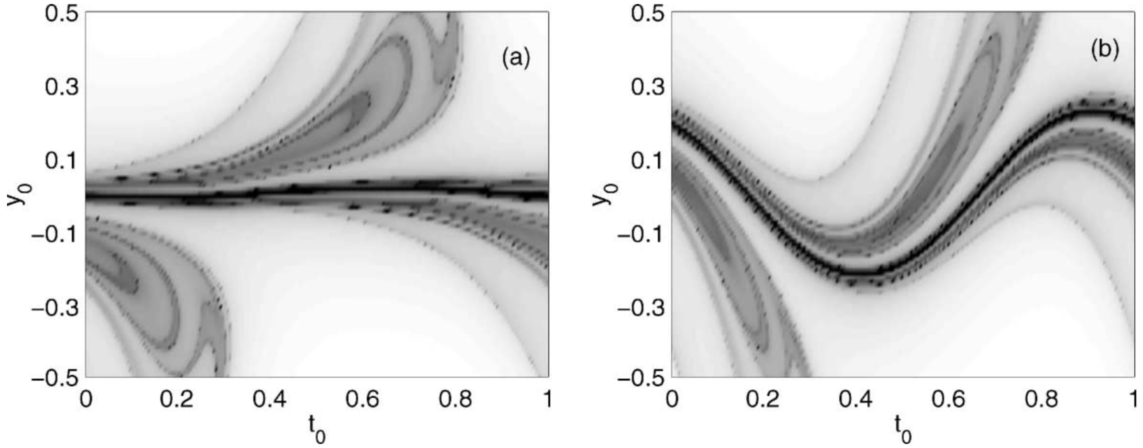

Fig. 5. Delay plots for the initial condition $x_0 = -5$, where δt is represented at different colours (a darker colour indicates a higher value) over a grid in which the horizontal axis represents the different starting times and the vertical axis represents the different values of y_0 that have been considered. Two cases are considered: (a) static cylinder, (b) oscillating cylinder with the same frequency as the frequency of the vortices and with an amplitude of 0.2 length units.

5. Conclusions

The fluid flow past a fixed and static cylinder is a paradigm of pattern formation and transient chaos, and it has deserved much attention by directly solving the corresponding Navier–Stokes equations, and also from the chaotic advection perspective. Our approach here has been to analyze under which circumstances we can control the transient chaos at the wake of the cylinder precisely from the perspective of non-linear dynamics. In order to apply a certain control technique which might, as a consequence, reduce the chaos present in the wake of the cylinder, we need to apply certain action on the cylinder. We have considered here two options of acting on the cylinder. From the previous ideas, we have adapted the model of the stream function, initially defined for a fluid flow past a static cylinder, for a certain value of the Reynold number, (see Ref. [5]), to two different situations in which the cylinder is not static. In the first one the cylinder moves in a rotating way and in the latter one the cylinder oscillates in a direction perpendicular to the incoming flow. By using standard methods of non-linear dynamics, we have analyzed how the chaoticity in the wake of the cylinder can be modified, or even suppressed for these situations. Several indicators of the complexity of the orbits have been used in order to check the effects on the dynamics of moving the cylinder. Moreover these results open new ways and perspectives at looking at the controlling the transient chaos of certain dynamical systems of invaluable applied interest.

Acknowledgements

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