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A generalized perturbed pendulum

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Abstract

The simple pendulum is a paradigm in the study of oscillations and other phenomena in physics and nonlinear dynamics. This explains why it has deserved much attention, from many viewpoints, for a long time. Here, we attempt to describe what we call a generalized perturbed pendulum, which comprises, in a single model, many known situations related to pendula, including different forcing and nonlinear damping terms. Melnikov analysis is applied to this model, with the result of general formulae for the appearance of chaotic motions that incorporate several particular cases. In this sense, we give a unified view of the pendulum.

1. Introduction

Since the time of Galileo [1], the pendulum has constituted a physical object, fascinating physicists and becoming one of the paradigms in the study of physics and natural phenomena. In the framework of nonlinear dynamics, there is no doubt that the pendulum is one of the objects that have deserved more attention in modelling all kind of phenomena related to oscillations, bifurcations and chaos. Its paradigmatic importance in mathematics has been also pointed out [2]. Its interest derives not only from its intrinsic value as a notable example to test and search for new phenomena, but also from its wide range of applicability.

From the theoretical point of view, its study may be considered of fundamental interest, where new results appear once in a while (for example, on the stability of pendula, following the theorem proved by Acheson [3–6]), and all possible combinations of pendula, such as the double pendulum [7,8], coupled pendula or even networks of pendula are used from very different perspectives and approaches [9–11]. One example of an interesting new result applied to this system refers to the Wada property, which was thoroughly studied for the forced pendulum in [12,13], and has to do with the unpredictability of the system, even when it has simple periodic attractors. On the other hand, very many physical phenomena may be modelled as pendula. This is because many oscillatory problems may be reduced in some way to the equations of the pendulum. One could argue that, as a kind of oscillatory unit, it may be found almost everywhere where oscillations occur. Apart from the familiar cases which appear in mechanics, it has been used to model a charged particle inside an electric field with applications to nuclear reactors and plasmas, and even as a universal model for nonlinear resonance [14]. Other fields of application, for example, are the Josephson superconducting unions [15–17], modelling of structural and electronic properties in condensed matter physics [18] and in celestial mechanics, especially related to the stability of the solar system [19], just to mention a few examples. A good source of examples mostly related to mechanical engineering and mechanics are found in the book by Moon [20]. Another reference, which is basically dedicated to many phenomena associated to pendula, including many applications

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to plasma physics, is the book by Sagdeev et al. [21]. Also, in problems related to engineering and control, the study of the pendulum is of much importance. For a textbook in which the basic characteristics of chaos are introduced with the help of the simple pendulum (see [22]).

The present work attempts to give a unified view of different known aspects of the pendulum when suffering distinct external perturbations. In particular we are interested in considering the general case of the plane simple pendulum, whose pivot is subjected to having different motions on the plane. For the case of the spherical pendulum, that is not included in this work (see [23–25]). In principle, our main interest lies in the possibility to integrating, in one single model, several known particular situations, such as the familiar forced pendulum, the harmonically forced pendulum [26–28], the vertically forced pendulum and combinations of them, such as the rotating pendulum. Moreover, we introduce in the expression of the generalized pendulum nonlinear damping terms, which in spite of its interest for many practical purposes, they are seldom used. All these models have been studied separately by many authors, but a general scheme as the one we are offering here is clearly lacking. Besides the forced pendulum, perhaps the case which has deserved more attention in the literature is the pendulum with a vertically oscillating pivot. This system was apparently first studied by Stephenson in 1908 [29] and somewhat later, in the twenties, by van der Pol and Strutt [30]. A good treatment of the inverted pendulum may be found in [31,32]. One outstanding interest in it relies upon its stability properties (see for instance [33,34] and the many references therein).

Once this generalized expression of the equations of motion of the simple pendulum, which we call a *generalized perturbed pendulum*, is obtained, different possible avenues of further study may be open. The strategy that we have followed here is based on the approach given by the Melnikov method, which typically applies to continuous dynamical systems. This method gives some conditions for the chaotic motion of these dynamical systems, which basically are related to the topological behavior of the invariant manifolds associated to hyperbolic saddle points in phase space. As a natural consequence of the use of this method to the generalized perturbed pendulum, some formulae related to the chaotic behavior (homoclinic chaos) of the pendulum are given, which comprise most particular cases that one may have taken into consideration.

2. Equations of motion of the generalized perturbed pendulum

Here we attempt to give a general formulation of the simple pendulum, where different forcing and damping terms are included in a single expression, with the aim of offering an overview of various situations that a pendulum may have and portrait all of them in a common framework. From this perspective, several familiar cases including external perturbations appear in a natural way, as particular cases of this generalized equation.

A simple mathematical pendulum is modelled by a bob of mass m, hanging at the end of a wire of length l and fixed to a supporting point O (see Fig. 1), swinging to and fro in a vertical plane.

The equations of motion are straightforward to obtain using Lagrangian or Newtonian methods. For its simplicity, we show here the pendulum equations using Newtonian methods. In this framework it is much more intuitive to visualize the forces acting on the system, providing a more clearer physical picture of the dynamics of the pendulum, even though other general formulations are possible. In this context Fig. 1 shows the force diagram of the simple pendulum,

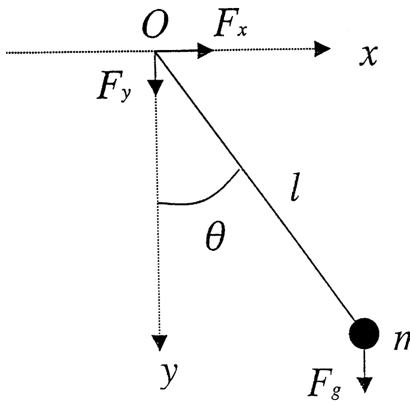

Fig. 1. The application of Newton's law to the force system shown in this figure results in the equation of motion of a generalized simple pendulum whose pivot may move on the plane.

whose pivot is situated at the coordinate (x_0, y_0), assuming a rectangular coordinate system. The coordinates of the bob in a fixed frame of reference are given by $(x, y) = (x_0 + l\sin\theta, y_0 + l\cos\theta)$. The equation of momenta of the forces with respect to the position of the bob is given by

$$F_x l\cos\theta - F_y l\sin\theta = 0, \tag{1}$$

where F_x and F_y are the components of the force acting on the pivot. The simple application of Newton's law gives

$$\begin{aligned} m\ddot{x} &= F_x, \\ m\ddot{y} &= F_y + mg, \end{aligned} \tag{2}$$

and introducing the coordinates of the bob with respect to the origin into the last equation, we obtain

$$\begin{aligned} \ddot{x} &= \ddot{x}_0 - l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta, \\ \ddot{y} &= \ddot{y}_0 - l\dot{\theta}^2\cos\theta - l\ddot{\theta}\sin\theta. \end{aligned} \tag{3}$$

Now, substituting Eq. (3) into Eq. (2), and then into Eq. (1), we get the following general equation of motion for the simple pendulum

$$l\ddot{\theta} + g\sin\theta + \ddot{x}_0\cos\theta - \ddot{y}_0\sin\theta = 0, \tag{4}$$

where it is assumed that the functions $\ddot{x}_0(t)$ and $\ddot{y}_0(t)$ are known (as a matter of fact they represent the equations of motion of the supporting point) and the coordinate θ is defined, as usual, in the interval $-\pi \leqslant \theta \leqslant \pi$. This equation can be also obtained using Lagrangian methods by simply considering the Lagrange function

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy = \tfrac{1}{2}m(\dot{x}_0^2 + \dot{y}_0^2) + ml\dot{\theta}(\dot{x}_0\cos\theta - \dot{y}_0\sin\theta) + \tfrac{1}{2}ml^2\dot{\theta}^2 + mgy_0 + mgl\cos\theta. \tag{5}$$

In principle, the generalized Eq. (4) represent the equation of the pendulum without dissipation and for any possible motion of the pivot on a plane. When dissipation is taken into account, traditionally dissipative forces are introduced ad hoc in the equations of motion, although a Lagrangian and Hamiltonian formalism for the case of linear damping is known [35]. The Lagrangian and Hamiltonian functions when linear dissipation is considered are, of course, time-dependent, and obviously do not have the same meaning as for conservative systems. The phenomenological model of the dissipative forces mostly appearing in the literature is assumed to be linear. However, and in order to offer a more general view of the equations of the pendulum, we will introduce here nonlinear damping terms. These terms have been used for several engineering applications, such as ship dynamics and vibration engineering (see for instance [36–38] and other references therein). One of the reasons why nonlinear damping in engineering and other applied sciences is important stems from the fact that it can be used as an effective passive control strategy to suppress various instabilities. Moreover, different effects of nonlinear damping on the dynamics of some nonlinear oscillators, including erosion of fractal basins and how they affect the routes to chaos, among others, have been shown recently. The nonlinear dissipative forces we consider are strictly proportional to the Nth power of the velocity, and consequently we use a general polynomial function of Nth degree of the form

$$\sum_{p=0}^{N} b_p\dot{\theta}|\dot{\theta}|^{p-1}, \tag{6}$$

where $b_p \geqslant 0$ are the nonlinear damping coefficients. Notice that the absolute value of $\dot{\theta}$ is needed to safeguard the fact that the dissipative force must be contrary to the motion of the system. Hence the general equation of motion for the nonlinearly damped pendulum is

$$l\ddot{\theta} + g\sin\theta + \ddot{x}_0\cos\theta - \ddot{y}_0\sin\theta + \sum_{p=0}^{N} b_p\dot{\theta}|\dot{\theta}|^{p-1} = 0. \tag{7}$$

The usual linear damping term is obtained by taking $b_1 > 0$ and $b_p = 0$, for $p \neq 1$. As it was mentioned earlier, in this equation we include the fact that the supporting point is movable and this will be used later to introduce, in a natural way, different forms of driving on the pendulum.

Furthermore, in the case that the supporting point is at rest, if we want some oscillations to be maintained, we need to introduce any other kind of perturbation to the system. A balance of energy is needed, so that the energy which is lost by the dissipation should be balanced by external sources if we want the oscillations to be sustained. This can be done by means of a periodic force $F(t)$ acting directly on the bob. Including ad hoc this kind of forcing in our model for a generalized simple pendulum we get

$$\ddot{\theta} + \sum_{p=0}^{N} \alpha_p \dot{\theta} |\dot{\theta}|^{p-1} + \omega_0^2 \sin\theta + f_1(t)\cos\theta + f_2(t)\sin\theta = f_3(t), \tag{8}$$

in which we have defined the natural frequency ω_0 in the usual way as $\omega_0^2 = g/l$ and the damping coefficients α_p as $\alpha_p = b_p/l$. Moreover, the forcing terms $f_1(t)$, $f_2(t)$ and $f_3(t)$, are defined as $f_1(t) = \ddot{x}_0(t)/l$, $f_2(t) = -\ddot{y}_0(t)/l$, $f_3(t) = F(t)/l$.

This last Eq. (8) describes what we call the generalized perturbed pendulum, including a rather general nonlinear damping term and several possible driving forces acting on it. To clarify this, we will illustrate in the following some basic particular examples of this general case, in some sense the elementary cases and thus the more familiar cases of driven pendula, which have been extensively studied in the literature.

1. *Pendulum with horizontal oscillating support*. The pivot of the pendulum is subjected to a harmonic horizontal displacement (see Fig. 2) of amplitude $l\varepsilon_1$ and frequency ω_1. This means that $f_1(t) = \varepsilon_1 \cos\omega_1 t$ and $f_2(t) = f_3(t) = 0$. In the usual case in which only linear dissipative forces are considered, the equations of motion are given by

$$\ddot{\theta} + \alpha_1\dot{\theta} + \omega_0^2 \sin\theta + \varepsilon_1 \cos\omega_1 t \cos\theta = 0. \tag{9}$$

A system like this one was used in order to analyze its chaotic properties and its bifurcations in [39,40]. It appears commonly in the fields of nonlinear vibrations and robotics.

2. *Pendulum with vertical oscillating support*. The pivot of the pendulum is subjected to a harmonic vertical displacement (see Fig. 3) of amplitude $l\varepsilon_2$ and frequency ω_2. Since the support oscillates harmonically in the vertical direction, this means that $f_2(t) = \varepsilon_2 \cos\omega_2 t$, and $f_1(t) = f_3(t) = 0$. For linear dissipative forces we get,

$$\ddot{\theta} + \alpha_1\dot{\theta} + \omega_0^2 \sin\theta + \varepsilon_2 \cos\omega_2 t \sin\theta = 0. \tag{10}$$

Note that this is in fact the equation of motion of the familiar inverted pendulum. A similar equation has been extensively used by many authors, among them we can refer to [41–51].

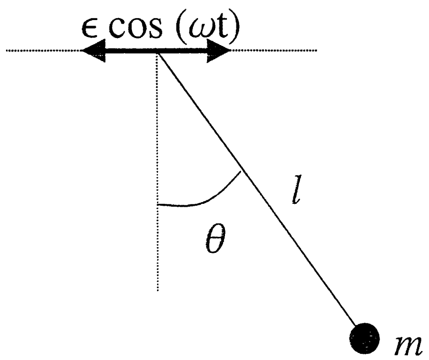

Fig. 2. The figure shows a simple pendulum whose supporting point oscillates harmonically in the horizontal direction.

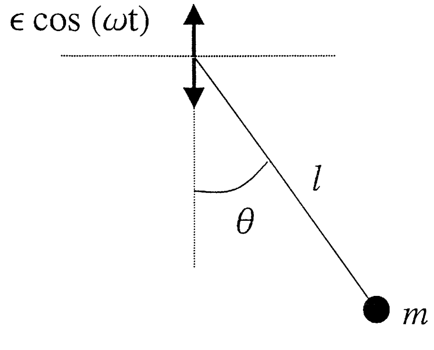

Fig. 3. The figure shows a simple pendulum whose supporting point oscillates harmonically in the vertical direction. Sometimes it is referred to as the parametrically excited pendulum.

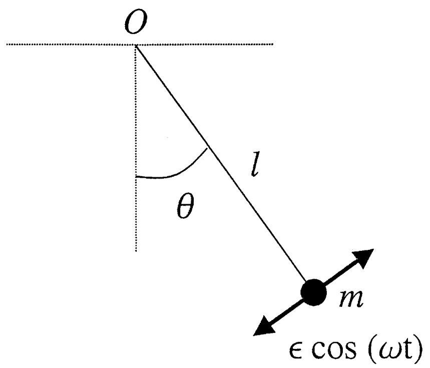

Fig. 4. A typical pendulum driven by an external harmonic function acting directly on the bob with the oscillating support at rest.

3. *Pendulum driven by an external force acting on the bob*. The harmonic perturbation is acting directly on the bob (see Fig. 4), so we have $f_3(t) = \varepsilon_3 \cos \omega_3 t$, and $f_1(t) = f_2(t) = 0$ and consequently the equation of motion is

$$\ddot{\theta} + \alpha_1 \dot{\theta} + \omega_0^2 \sin \theta = \varepsilon_3 \cos \omega_3 t, \tag{11}$$

which corresponds to the familiar damped and perturbed pendulum, with fixed pivot.

The inclusion of the perturbation in the three cases makes it possible to sustain oscillations, and hence the possibility of periodic solutions, and eventually for certain parameters the appearance of chaotic solutions. These three cases are the most familiar, and perhaps the most elementary, that one can obtain from the generalized expression, although other versions may appear by considering different combinations of the terms.

3. Melnikov method for a generalized simple pendulum

Once we have described a general model for the pendulum, including different driving forces and nonlinear damping terms, our interest now is to apply Melnikov method in order to obtain general expressions for the critical parameters for the occurrence of horseshoe dynamics. The Melnikov analysis yields the threshold values of the parameters for which homoclinic intersections occur. For details concerning Melnikov method see [53]. A nice energy interpretation of the method is provided by Ketema [54]. As is well known, this technique is a first-order perturbative method which gives the condition for the crossing of the stable and unstable manifolds; that is to say, this does not mean that what we have is *permanent chaos*, but a horseshoe type dynamics associated with the phenomenon of *transient chaos*. Although the chaos does not manifest itself in the form of permanent chaos, it does in terms of the fractal basin boundaries, as it was shown by Moon and Li [55].

From this perspective, we consider here a generalized pendulum

$$\ddot{\theta} + \sum_{p=0}^{N} \alpha_p \dot{\theta} |\dot{\theta}|^{p-1} + \omega_0^2 \sin \theta + \varepsilon_1 f_1(t) \cos \theta + \varepsilon_2 f_2(t) \sin \theta = \varepsilon_3 f_3(t), \tag{12}$$

in which we assume that the damping coefficients α_p and amplitudes ε_i are small enough, so we can compute the Melnikov integral for this system, considering all extra terms as Hamiltonian perturbations to the simple pendulum with equation

$$\ddot{\theta} + \omega_0^2 \sin \theta = 0. \tag{13}$$

The Hamiltonian function for the variables $(\theta, \dot{\theta}) \in [-\pi, \pi] \times R$ is

$$H(\theta, \dot{\theta}) = \tfrac{1}{2}\dot{\theta}^2 - \omega_0^2 \cos \theta. \tag{14}$$

The phase space of the pendulum is 2π-periodic in θ with hyperbolic saddles situated in $(\pm\pi, 0)$ and an elliptic centre in $(0, 0)$. There are three kind of orbits for the unperturbed pendulum: rotations, oscillations and separatrix motion. The rotations are unbounded motions and correspond to orbits with high energy moving clockwise or counter-clockwise, the oscillations are the bounded ones and they never reach the elliptic centre; finally, the separatrix motion corresponds to an oscillation of infinite period. We can identify both hyperbolic saddles and consider a cylindrical phase space, in such a way that we may have homoclinic connections and motions, which correspond to the motions in the separatrix. The solutions for the oscillating orbits can be expressed in terms of the Jacobi elliptic functions as [43,45]

$$(\theta(t),\dot{\theta}(t)) = (2k\,\mathrm{cn}(\Omega t,k)\,\mathrm{dn}(\Omega t,k), 2k\,\mathrm{cn}(\Omega t,k)), \tag{15}$$

where the functions $\mathrm{cn}(\Omega t,k)$ and $\mathrm{dn}(\Omega t,k)$ are Jacobi elliptic functions [52] of frequency $\Omega(k)$ and elliptic modulus k. Sometimes the elliptic parameter m is used instead where $m=k^2$. The solutions for the homoclinic orbit are obtained by simply taking the limit $k\to 1$ for the elliptic parameter in the Eq. (15). The solutions are given by setting $H(\theta,\dot{\theta})=h$, since the energy is conserved. If $h=\omega_0^2$, then we have a pair of homoclinic solutions given by

$$\theta_0^{\pm}(t) = \pm 2\arctan[\sinh(\omega_0 t)] = \pm 2\tanh\omega_0 t\,\mathrm{sech}\,\omega_0 t, \tag{16}$$

$$\dot{\theta}_0^{\pm}(t) = \pm 2\omega_0\,\mathrm{sech}(\omega_0 t), \tag{17}$$

subject to the initial conditions $(\theta_0^{\pm}(0),\dot{\theta}_0^{\pm}(0)) = (0,\pm 2\omega_0)$.

Now, applying the Melnikov theory to the general case we are studying, a generalized Melnikov function can be written as

$$\begin{aligned}M^{\pm}(t_0) = &-\sum_{p=0}^{N}\alpha_p\int_{-\infty}^{+\infty}|\dot{\theta}_0^{\pm}(t)|^{p+1}\,\mathrm{d}t - \varepsilon_1\int_{-\infty}^{+\infty}\dot{\theta}_0^{\pm}(t)\cos(\theta_0^{\pm}(t))f_1(t+t_0)\,\mathrm{d}t\\ &-\varepsilon_2\int_{-\infty}^{+\infty}\dot{\theta}_0^{\pm}(t)\sin(\theta_0^{\pm}(t))f_2(t+t_0)\,\mathrm{d}t + \varepsilon_3\int_{-\infty}^{+\infty}\dot{\theta}_0^{\pm}(t)f_3(t+t_0)\,\mathrm{d}t.\end{aligned} \tag{18}$$

From the expression of the homoclinic orbits Eqs. (16) and (17), it is quite straightforward to obtain the formulae

$$\begin{aligned}\sin(\theta_0(t)) &= 2\sin A\cos A = 2\tanh(\omega_0 t)\,\mathrm{sech}(\omega_0 t),\\ \cos(\theta_0(t)) &= \cos^2 A - \sin^2 A = \mathrm{sech}^2(\omega_0 t) - \tanh^2(\omega_0 t),\end{aligned} \tag{19}$$

where $A=\arctan(\sinh(\omega_0 t))$, which are quite useful for the computations. Then

$$\begin{aligned}M^{\pm}(t_0) = &-\sum_{p=0}^{N}2^{p+1}\omega_0^p\alpha_p\int_{-\infty}^{+\infty}\mathrm{d}t\,\mathrm{sech}^{p+1}t \mp 2\omega_0\varepsilon_1\int_{-\infty}^{+\infty}\mathrm{d}t\,f_1(t+t_0)\big(2\,\mathrm{sech}^3(\omega_0 t) - \mathrm{sech}(\omega_0 t)\big)\\ &\mp 4\omega_0\varepsilon_2\int_{-\infty}^{+\infty}\mathrm{d}t\,f_2(t+t_0)\tanh(\omega_0 t)\,\mathrm{sech}^2(\omega_0 t) \pm 2\omega_0\varepsilon_3\int_{-\infty}^{+\infty}\mathrm{d}t\,f_3(t+t_0)\,\mathrm{sech}(\omega_0 t),\end{aligned} \tag{20}$$

that can be taken to be a generalized Melnikov function for the simple pendulum. This expression comprises in a compact way a lot of particular results that can be found in the literature. Obviously this generalized result comprises the results of authors who have analyzed particular cases, such as Koch and Leven [45] and Huilgol et al. [57], and Ravindra and Mallik [58,59], among others.

4. Application to particular cases

Consider now the most used case in which all the driving forces $f_1(t)$, $f_2(t)$ and $f_3(t)$ are periodic. Then, we can write $f_i(t)=\cos(\omega_i t+\delta_i)$ $(i=1,2,3)$, and interpret ε_i, ω_i and δ_i as the amplitudes, frequencies and initial phases of the perturbations, respectively.

In this case, the generalized Melnikov function (20) reduces to

$$\begin{aligned}M^{\pm}(t_0) = &-\sum_{p=0}^{N}2^{p+1}\omega_0^p\alpha_p B\left(\frac{1}{2},\frac{p+1}{2}\right) \mp 2\pi\varepsilon_1\left(\frac{\omega_1}{\omega_0}\right)^2\mathrm{sech}\left(\frac{\pi\omega_1}{2\omega_0}\right)\cos(\omega_1 t_0+\delta_1)\\ &\pm 2\pi\varepsilon_2\left(\frac{\omega_2}{\omega_0}\right)^2\mathrm{csch}\left(\frac{\pi\omega_2}{2\omega_0}\right)\sin(\omega_2 t_0+\delta_2) \pm 2\pi\varepsilon_3\,\mathrm{sech}\left(\frac{\pi\omega_3}{2\omega_0}\right)\cos(\omega_3 t_0+\delta_3),\end{aligned} \tag{21}$$

in which we have used some basic integrals that are tabulated at the end of this article in the Appendix A. In this expression, $B(m,n)$ is the Euler Beta function, defined in terms of the Euler Gamma function as $B(m,n)=\Gamma(m)\Gamma(n)/\Gamma(m+n)$ [56]. Since the Melnikov function is related to the distance between the stable and the unstable manifolds associated with the hyperbolic fixed point, when destroyed by the perturbation, this implies that when this function has simple zeros, there is a critical set of parameters for which homoclinic tangles intersect. For this purpose, and taking into account the behavior of the trigonometric functions, it is enough to consider only the positive branch. Thus, in the following, we study this Melnikov function $M(t_0) = M^+(t_0)$ for different particular cases. Some of them shows very interesting properties.

4.1. Critical parameters for the case of one forcing term

The first case that we are interested approaching here is the case when only one forcing term $f_i(t)$ is acting on the simple pendulum. A particular example of this approach is a damped pendulum in which the supporting point moves harmonically along the horizontal direction, so this implies $\varepsilon_2 = \varepsilon_3 = 0$. Moreover, we assume that the dissipative force is linear in the velocity, $\alpha_p = 0$ for $p \neq 1$, and for simplicity the initial time is chosen in such a way that the initial phase is $\delta_1 = 0$. The idea is to apply the Melnikov analysis and to analytically compute the positive branch of the Melnikov function in Eq. (21), which for this particular case is reduced to

$$M(t_0) = -8\omega_0\alpha_1 - 2\pi\varepsilon_1\left(\frac{\omega_1}{\omega_0}\right)^2 \operatorname{sech}\left(\frac{\pi\omega_1}{2\omega_0}\right)\cos\omega_1 t_0. \tag{22}$$

Then the set of critical parameters for which the invariant manifolds intersect is given by the condition that this function has a simple zero. As a result, the critical forcing amplitude ε_{1c} is given by

$$\varepsilon_{1c} = \frac{4\omega_0^3\alpha_1}{\pi\omega_1^2}\cosh\left(\frac{\pi\omega_1}{2\omega_0}\right). \tag{23}$$

In order to gain an understanding of this behavior, it is convenient and useful to define the ratio $R(\omega_0, \omega_1) = \varepsilon_{1c}/\alpha_1$ between the critical forcing amplitude and the damping coefficient, which is given by the expression

$$R(\omega_0, \omega_1) = \frac{4\omega_0^3}{\pi\omega_1^2}\cosh\left(\frac{\pi\omega_1}{2\omega_0}\right). \tag{24}$$

To have a visual information on this ratio, we have plotted in Fig. 5 its dependence with respect to the forcing frequency ω_1 for the values $\omega_0 = 0.5$, $\omega_0 = 1$ and $\omega_0 = 1.5$ of the natural frequency. The observation of this figure shows that for the range of forcing frequencies up to approximately 1.3, the Melnikov ratio increases as the natural frequency ω_0 increases. This is not, however, a general result, since it changes for higher values of the forcing frequency as shown in the figure. From our point of view, the shape of the curves is more important, and, in particular, the presence of a local minimum for every value of ω_0. This means that the system has different sensitivities to the onset of homoclinic chaos depending on the forcing frequency of the pivot. Also, we can infer from Fig. 5 that the region in which the system is

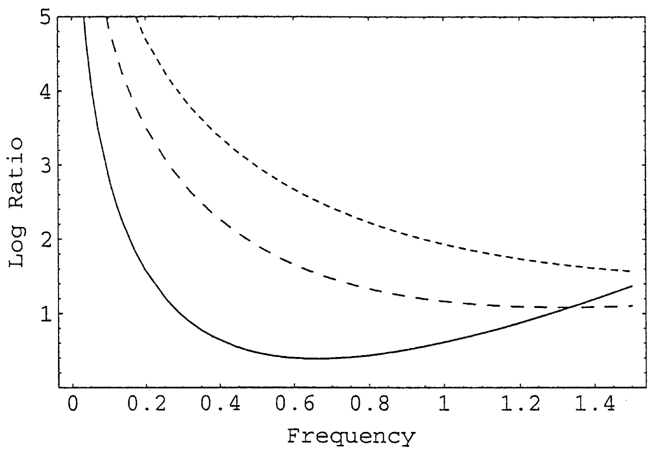

Fig. 5. Plot of the natural logarithm of the Melnikov ratio between the critical parameters versus the frequency of the driving force for $\omega_0 = 0.5$ (—), $\omega_0 = 1$ (– – –) and $\omega_0 = 1.5$ (- - -). The case shown corresponds to a damped pendulum in which the supporting point moves harmonically along the horizontal direction. For forcing frequencies up to 1.3, the Melnikov ratio increases as the natural frequency ω_0 increases. This behavior changes for higher values of the forcing frequency, which are not shown in this figure.

more sensitive to the onset of homoclinic chaos (the Melnikov ratio is lower), increases as ω_0 increases. This means that, in general, it is more difficult to find a region in which the system evolves in a nonchaotic regime when ω_0 is raised, for any given forcing frequency ω_1. In other words, suppression of chaos is easier when the natural frequency is lower. In these cases of low ω_0 we only need to slightly modify the forcing frequency ω_1 from the one that minimizes the Melnikov ratio. For higher values of ω_0, the modification of ω_1 must be stronger.

Another interesting result appears when we consider nonlinear damping terms. For example, if we introduce a cubic damping term to the previous example, then the Melnikov function is

$$M(t_0) = -8\omega_0\alpha_1 - \frac{64}{3}\omega_0^3\alpha_3 - 2\pi\varepsilon_1\left(\frac{\omega_1}{\omega_0}\right)^2 \operatorname{sech}\left(\frac{\pi\omega_1}{2\omega_0}\right)\cos\omega_1 t_0, \tag{25}$$

in which α_3 is the cubic damping coefficient. The critical value of the forcing amplitude ε'_{1c} becomes

$$\varepsilon'_{1c} = \frac{4\omega_0^3\alpha_1 + 32/3\omega_0^5\alpha_3}{\pi\omega_1^2}\cosh\left(\frac{\pi\omega_1}{2\omega_0}\right) = \varepsilon_{1c}\left(1 + \frac{8\omega_0^2\alpha_3}{3\alpha_1}\right). \tag{26}$$

This clearly shows that the introduction of the nonlinear damping terms has the effect that the critical value of the forcing amplitude increases linearly with the nonlinear damping coefficient. As is easily inferred from this expression, the positions of the local minima are not modified by the inclusion of nonlinear damping terms of the type considered here. Also, it should be noticed that the critical value of the forcing amplitude has a stronger dependence on the natural frequency than in the case when only linear damping terms are used. Furthermore, in terms of sensitivity, the inclusion of nonlinear damping terms makes more difficult the onset of homoclinic chaos in the system. Some interesting results concerning these ideas are illustrated in [37].

4.2. Critical parameters for the case of two forcing terms

Now we consider the case when two forcing terms $f_i(t)$ are acting on the simple pendulum. We may have three different possibilities.

(a) The supporting point of the linearly damped simple pendulum moves harmonically in both the horizontal and vertical directions ($\varepsilon_3 = 0$). Then the equation of motion reads

$$\ddot{\theta} + \alpha_1\dot{\theta} + \omega_0^2\sin\theta + \varepsilon_1\cos(\omega_1 t + \delta_1)\cos\theta + \varepsilon_2\cos(\omega_2 t + \delta_2)\sin\theta = 0. \tag{27}$$

Note that the supporting point is subjected to a superposition of two orthogonal harmonic oscillators, so the resultant trajectory of the motion of this point depends strongly on the value of the frequency ratio ω_1/ω_2 and the phase difference $\delta_1 - \delta_2$. A very well known mechanical example is the rotating pendulum, in which the supporting point presents circular polarization. This model was also considered in [60]. In this case, $\varepsilon_1 = \varepsilon_2$, $\omega_1 = \omega_2$ and $\delta_1 - \delta_2 = \pi/2$. The Melnikov function then appears as

$$M(t_0) = -8\omega_0\alpha_1 - 2\pi\varepsilon_1\left(\frac{\omega_1}{\omega_0}\right)^2\left\{\operatorname{sech}\left(\frac{\pi\omega_1}{2\omega_0}\right) + \operatorname{csch}\left(\frac{\pi\omega_1}{2\omega_0}\right)\right\}\cos(\omega_1 t_0 + \delta_1). \tag{28}$$

The ratio between the critical forcing amplitude and the damping coefficient is

$$R(\omega_0, \omega_1) = \frac{4\omega_0^3}{\pi\omega_1^2}\left\{\operatorname{sech}\left(\frac{\pi\omega_1}{2\omega_0}\right) + \operatorname{csch}\left(\frac{\pi\omega_1}{2\omega_0}\right)\right\}^{-1}. \tag{29}$$

In Fig. 6, a plot of the ratio $R(\omega_0, \omega_1) = \varepsilon_c/\alpha_1$ versus the forcing frequency ω_1 for the values $\omega_0 = 0.5$, $\omega_0 = 1$ and $\omega_0 = 1.5$ of the natural frequency is given. As we have done the same study as in Fig. 5, it is easy to see what is the difference between these two cases: the introduction of the vertical harmonic forcing in circular polarization with the horizontal forcing makes the intersection of the tangles to occur before, i.e., the value of the critical parameter ε_c is lower than in the previous case and then the sensitivity of the system to the onset of chaos is increased. But it is important to note that this is a property of the particular case of circular polarization, not of any superposition of two orthogonal harmonic oscillators in the supporting point. In particular the expression Eq. (29) may result much more complicated for more general cases.

(b) The supporting point of the linearly damped simple pendulum moves harmonically in the horizontal direction and the bob is subjected to a harmonic forcing ($\varepsilon_2 = 0$). Then the equation of motion is

$$\ddot{\theta} + \alpha_1\dot{\theta} + \omega_0^2\sin\theta + \varepsilon_1\cos(\omega_1 t + \delta_1)\cos\theta = \varepsilon_3\cos(\omega_3 t + \delta_3), \tag{30}$$

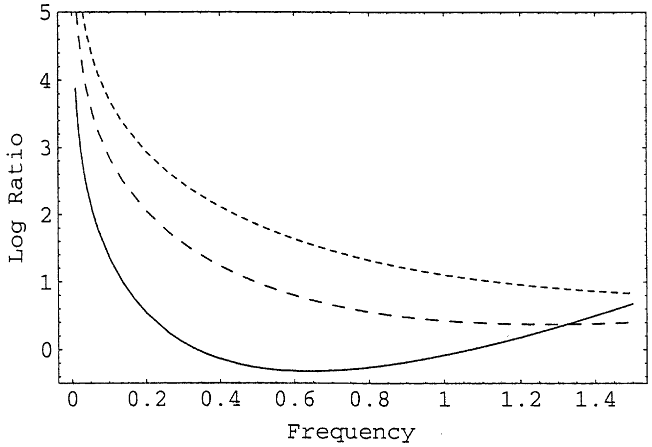

Fig. 6. Natural logarithms of the Melnikov ratio between critical parameters versus frequency of the forcing for $\omega_0 = 0.5$ (—), $\omega_0 = 1$ (– – –) and $\omega_0 = 1.5$ (- - -). Here the supporting point of the pendulum oscillates harmonically in the horizontal and vertical directions. This figure is rather similar to the previous one, and shows that the ratio is smaller due to the introduction of two forcing terms in circular polarization. Consequently, the value of the critical parameter ε_c is lower than in the previous case.

and the Melnikov function

$$M(t_0) = -8\omega_0\alpha_1 - 2\pi\varepsilon_1\left(\frac{\omega_1}{\omega_0}\right)^2 \operatorname{sech}\left(\frac{\pi\omega_1}{2\omega_0}\right)\cos(\omega_1 t_0 + \delta_1) + 2\pi\varepsilon_3 \operatorname{sech}\left(\frac{\pi\omega_3}{2\omega_0}\right)\cos(\omega_3 t_0 + \delta_3). \tag{31}$$

A three-dimensional plot of this function versus the external frequencies ω_1 and ω_3 is shown in Fig. 7. The parameters we are using are set to $\delta_1 = \delta_3 = 0$, $\omega_0 = 1$, $\alpha_1 = 0.1$ and $\varepsilon_1 = \varepsilon_3 = 0.1$, for the time $t_0 = 1$. Our analysis shows that the Melnikov function smoothly decreases with the frequency ω_3 for a fixed horizontal frequency ω_1. However, if we fix ω_3 there is a local maximum and a local minimum in the range of frequencies considered.

A very interesting case appears when both forcing terms have the same amplitude ($\varepsilon_1 = \varepsilon_3$), and they are in phase ($\delta_1 = \delta_3$) and in resonance ($\omega_1 = \omega_3 = \omega_0$). In this case, the Melnikov function does not depend on t_0 and, consequently, the possible chaotic behavior that the horizontal forcing may introduce is suppressed by the external forcing

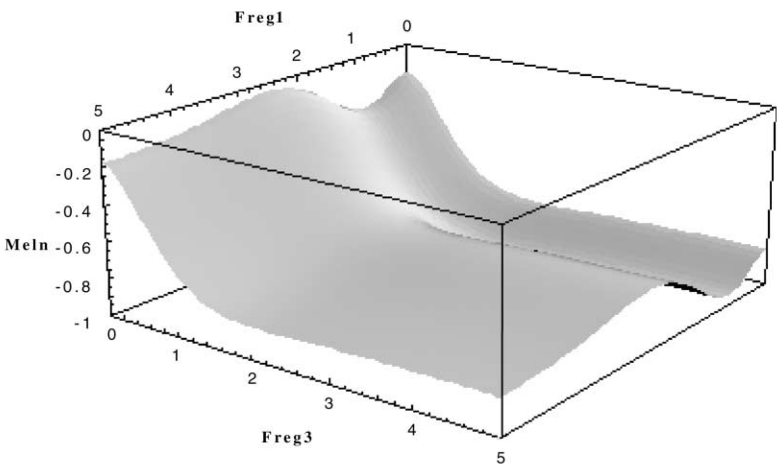

Fig. 7. A three-dimensional plot of the Melnikov function versus the external frequencies ω_1 and ω_3 for the case of a pendulum whose supporting point oscillates harmonically in the horizontal direction and the bob is subjected to a harmonic forcing. The figure shows the particular case for which the parameters are set to $\delta_1 = \delta_3 = 0$, $\omega_0 = 1$, $\alpha_1 = 0.1$ and $\varepsilon_1 = \varepsilon_3 = 0.1$, and the time is $t_0 = 1$. The Melnikov function decreases smoothly with the frequency ω_3 for a fixed horizontal frequency ω_1. However if we fix ω_3 there is a local maximum and a local minimum in the range of frequencies considered.

term. This means that the Melnikov function is constant, and, hence, there is no possible nontrivial zero. This is indeed a way of controlling chaos via the nonfeedback mechanism [61]. In fact, the idea of modifying the onset of chaos by studying the Melnikov function of the newly perturbed system was already used in [62].

(c) The supporting point of the linearly damped simple pendulum moves harmonically in the vertical direction and the bob is subjected to a harmonic forcing ($\varepsilon_1 = 0$). Then

$$\ddot{\theta} + \alpha_1 \dot{\theta} + \omega_0^2 \sin\theta + \varepsilon_2 \cos(\omega_2 t + \delta_2) \sin\theta = \varepsilon_3 \cos(\omega_3 t + \delta_3), \tag{32}$$

and its associated Melnikov function is

$$M(t_0) = -8\omega_0\alpha_1 - 2\pi\varepsilon_2 \left(\frac{\omega_2}{\omega_0}\right)^2 \operatorname{csch}\left(\frac{\pi\omega_2}{2\omega_0}\right) \sin(\omega_2 t_0 + \delta_2) + 2\pi\varepsilon_3 \operatorname{sech}\left(\frac{\pi\omega_3}{2\omega_0}\right) \cos(\omega_3 t_0 + \delta_3). \tag{33}$$

What we have mentioned in the previous case does not occur here, since it is not possible that the time-dependent terms cancel.

To investigate this case, take for example the values $\delta_2 = 0$, $\delta_3 = -\pi/2$, $\varepsilon_2 = \varepsilon_3$ and $\omega_2 = \omega_3$. Then the ratio between the critical value of the external forcing and the damping coefficient is given by

$$R(\omega_0, \omega_2) = \frac{4\omega_0}{\pi} \left\{ \operatorname{sech}\left(\frac{\pi\omega_2}{2\omega_0}\right) - \left(\frac{\omega_2}{\omega_0}\right)^2 \operatorname{csch}\left(\frac{\pi\omega_2}{2\omega_0}\right) \right\}^{-1}. \tag{34}$$

A plot of this ratio versus the forcing frequency ω_2 for the values $\omega_0 = 0.5$, $\omega_0 = 1$ and $\omega_0 = 1.5$ of the natural frequency is depicted in Fig. 8. The figure shows an asymptotic behavior for the resonant frequencies, suggesting that it is not possible to find chaotic solutions in these cases.

4.3. Critical parameters for the case of three forcing terms

As a last final example, we consider now the more general case when all the three harmonic forcing terms $f_i(t)$ are acting on the linearly damped pendulum. In this case, the equation of motion is given by Eq. (12) where $\alpha_p = 0$ for $p \neq 1$.

Of course, this general case comprises many particular situations. For this reason we have chosen to analyze the simplified case $\omega_1 = \omega_2 = \omega_3 = \omega$, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon$ and $\delta_1 = \delta_2 = \delta_3 = 0$, for which the Melnikov function is simpler. As a result, we obtain for this situation the following expression

$$M(t_0) = -8\omega_0\alpha_1 + 2\pi\varepsilon \left(1 - \frac{\omega^2}{\omega_0^2}\right) \operatorname{sech}\left(\frac{\pi\omega}{2\omega_0}\right) \cos(\omega t_0) + 2\pi\varepsilon \left(\frac{\omega^2}{\omega_0^2}\right) \operatorname{csch}\left(\frac{\pi\omega}{2\omega_0}\right) \sin(\omega t_0). \tag{35}$$

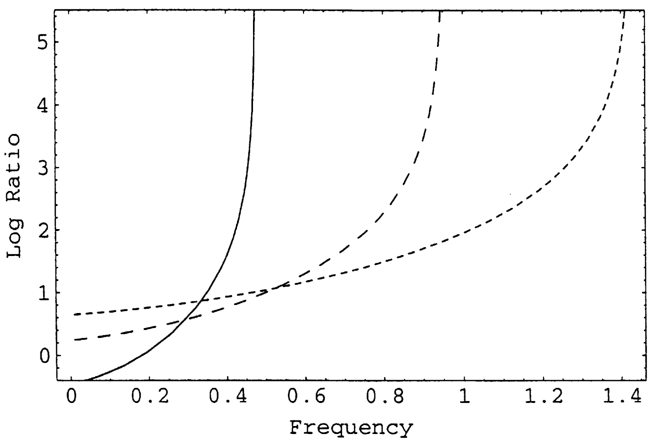

Fig. 8. Semilogarithmic plot of the Melnikov ratio versus the forcing frequency ω_2 for the values $\omega_0 = 0.5$ (—), $\omega_0 = 1$ (– – –) and $\omega_0 = 1.5$ (- - -). Notice the asymptotic behavior of this ratio for the resonant frequencies, which suggest that chaotic solutions in these cases are not possible to find. The case shown corresponds to a damped pendulum in which the supporting point moves harmonically in the vertical direction and the bob is subjected to a harmonic forcing.

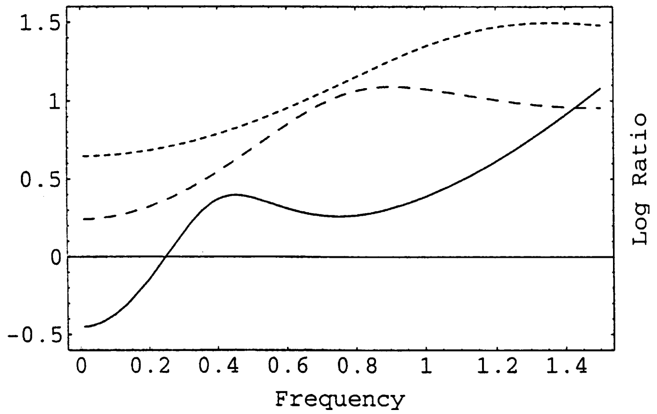

Fig. 9. Semilogarithmic plot of the Melnikov ratio versus the forcing frequency ω for the values $\omega_0 = 0.5$ (—), $\omega_0 = 1$ (– – –) and $\omega_0 = 1.5$ (- - -). This example represents a simplified version of the case when the three forcing terms are acting at the same time.

By simple inspection of this formula, it can be noted that there are two kinds of terms here, one of them affecting the sine function and the other one affecting the cosine function, something that does not occur in the previous cases under analysis. Since in the above examples there is only a kind of oscillating function, searching for the zeros of the Melnikov function is simpler. However, in the case we are analyzing here, this does not happen and we need to find first the time $\widetilde{t_0}$, which maximizes $M(t_0)$. This time $\widetilde{t_0}$ satisfies the condition

$$\tan(\omega\widetilde{t_0}) = \frac{\omega^2}{\omega_0^2 - \omega^2}\coth\left(\frac{\pi\omega}{2\omega_0}\right), \tag{36}$$

and the value of the Melnikov function at this point $\widetilde{t_0}$ is given by

$$M(\widetilde{t_0}) = -8\omega_0\alpha_1 + \frac{2\pi\varepsilon}{\omega_0^2}\sqrt{\omega^4\,\mathrm{csch}^2\left(\frac{\pi\omega}{2\omega_0}\right) + (\omega_0^2 - \omega^2)^2\,\mathrm{sech}^2\left(\frac{\pi\omega}{2\omega_0}\right)}. \tag{37}$$

It is important to recall that the previously computed value $M(\widetilde{t_0})$ represents the maximum of the Melnikov function and, thus, all we need to do is to set it to zero to find the corresponding critical parameters for which the invariant manifolds intersect. Once this computation is carried out, the ratio $R(\omega_0, \omega)$ between the critical value ε_c of the external amplitude ε and the damping coefficient α_1 is given by

$$R(\omega_0, \omega) = \frac{4\omega_0^3}{\pi\sqrt{\omega^4\,\mathrm{csch}^2\left(\frac{\pi\omega}{2\omega_0}\right) + (\omega_0^2 - \omega^2)^2\,\mathrm{sech}^2\left(\frac{\pi\omega}{2\omega_0}\right)}}. \tag{38}$$

Fig. 9 shows a plot of this ratio versus the forcing frequency ω for the natural frequency values $\omega_0 = 0.5$, $\omega_0 = 1$ and $\omega_0 = 1.5$. We have mainly analyzed frequencies on a small range up to 1.5. For this range of values, the Melnikov ratio increases with the natural frequency and shows a local maximum for values slightly smaller than the natural frequency. This means that when the pendulum is perturbed using frequencies in the vicinity of ω_0, one needs to increase the strength of the perturbation in order to get chaotic behavior, because the system tends to be locked to the external frequency by a mechanism of resonance, so avoiding the appearance of chaos. Note that this is exactly the opposite case to the situation described in Fig. 5, where the system sensitivity has a maximum near the natural frequency.

5. Discussion and conclusions

Application of Newton's law to the motion of the simple pendulum with nonlinear damping terms and incorporating several drivings leads to the equations of motion of what we call a generalized perturbed pendulum. This model is a paradigm for the study of many properties of continuous dynamical systems, with many applications to physical and technological problems.

We apply Melnikov method, which has proved useful in many practical cases to ascertain the chaotic responses of certain dynamical systems, to the generalized harmonically perturbed pendulum, resulting in general expressions that comprise particular cases. Among all the examples analyzed, an interesting particular situation appears when the supporting point of a linearly damped pendulum moves harmonically in the horizontal direction and the bob is subjected to a harmonic forcing, because there is a set of forcing parameters that removes the effect of the driving from the Melnikov function.

One of the strategies we have pursued here is analyzing the Melnikov ratio between the critical forcing amplitude and the damping coefficient for some specific examples. Its dependence with the natural and forcing frequencies has been determined for each case.

Finally, we would like to stress that even though most of the particular cases of this generalized perturbed pendulum have been studied separately by different authors, here we have attempted to offer a general scheme comprising all them and also showing general formulae that can be of further use, and that which might be extended to attack other problems in nonlinear dynamics.

Acknowledgements

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Appendix A

All the integrals used are

$$\int_{-\infty}^{+\infty} \mathrm{d}t\, \frac{\sinh^{\mu} t}{\cosh^{\nu} t} = B\left(\frac{\mu+1}{2}, \frac{\nu-\mu}{2}\right), \tag{A.1}$$

$$\int_{-\infty}^{+\infty} \mathrm{d}t\, \mathrm{sech}\, At \cos Bt = \frac{\pi}{A}\, \mathrm{sech}\left[\frac{\pi B}{2A}\right], \tag{A.2}$$

$$\int_{-\infty}^{+\infty} \mathrm{d}t\, \mathrm{sech}^2 At \tanh At \sin Bt = \frac{\pi B^2}{2A^3}\, \mathrm{csch}\left[\frac{\pi B}{2A}\right], \tag{A.3}$$

$$\int_{-\infty}^{+\infty} \mathrm{d}t\, \mathrm{sech}^3 At \cos Bt = \frac{\pi}{2A}\left\{1 + \frac{B^2}{A^2}\right\} \mathrm{sech}\left[\frac{\pi B}{2A}\right], \tag{A.4}$$

$$\int_{-\infty}^{+\infty} \mathrm{d}t\, \mathrm{sech}\, At \tanh^2 At \cos Bt = \frac{\pi}{2A}\left\{1 - \frac{B^2}{A^2}\right\} \mathrm{sech}\left[\frac{\pi B}{2A}\right]. \tag{A.5}$$

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