Basin Topology in Dissipative Chaotic Scattering

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Abstract

Chaotic scattering in open Hamiltonian systems under weak dissipation is not only of fundamental interest but also important for problems of current concern such as the advection and transport of inertial particles in fluid flows. Previous work using discrete maps demonstrated that nonhyperbolic chaotic scattering is structurally unstable in the sense that the algebraic decay of scattering particles immediately becomes exponential in the presence of weak dissipation. Here we extend the result to continuous-time Hamiltonian systems by using the Hénon-Heiles system as a prototype model. More importantly, we go beyond to investigate the basin structure of scattering dynamics. A surprising finding is that, in the common case where multiple destinations exist for scattering trajectories, Wada basin boundaries are common and they appear to be structurally *stable* under weak dissipation even when other characteristics of the nonhyperbolic scattering dynamics are not. We provide numerical evidence and a geometric theory for the structural stability of the complex basin topology.

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A fundamental issue in nonlinear dynamics is to understand the robustness of a phenomenon in physical situations where deviations from model assumptions may arise. Take, for example, scattering in Hamiltonian systems. In an idealized situation where the dynamics is conservative, particles coming from far away into a region of interaction must exit it in finite times, as no attracting sets are possible. Any dynamical invariant set in the scattering region must then be nonattracting. A regular invariant set, such as an unstable periodic orbit, gives rise to regular scattering, while chaotic scattering is the consequence of a nonattracting chaotic invariant set. In a realistic situation, factors such as weak dissipations and/or noise may be present. It is important to understand what dynamical phenomena may persist in the presence of various physical perturbations. Chaotic scattering has been studied for more than two decades because it is relevant to a host of areas in physics such as astrophysics, optics, fluid mechanics, nanophysics, etc. In most existing works on chaotic scattering, the underlying dynamical systems are assumed to be purely Hamiltonian. However, it is possible to conceive physical applications where this assumption may not be strictly valid. For instance, for chaotic scattering arising in the context of particle advection in hydrodynamical flows, the effect of weak dissipation can be important. This is so because the condition of incompressibility allows the problem to be casted in Hamiltonian dynamics as the particle velocities can be related to flow's stream function in a way that is completely analogous to the Hamilton's equations in classical mechanics. Real hydrodynamical flows cannot be perfectly incompressible, and the effects of inertia and finite mass of the particles advected by the flow are effectively those due to friction, or dissipation. The focus of our work is on the effect of weak dissipation on chaotic scattering in continuous-time Hamiltonian systems. A previous work addressed the topic but from the standpoint of the particle decay law. In addition, the models used were discrete-time maps. Here we investigate how the basin structure may be affected by dissipation and how physically measurable fractal dimensions may change. These issues are particularly relevant to nonhyperbolic scattering, where chaotic sets and Kol'mogorov-Arnol'd-Moser (KAM) tori coexist and dissipation can convert some KAM islands into attractors. Our main finding is that the complicated, fractal basin structures such as Wada basin boundaries typically persist under weak dissipation, despite metamorphic changes in the particle decay law. We expect this result to be useful as the basin structure associated with scattering dynamics is potentially experimentally accessible.

I. INTRODUCTION

In this paper we study the phenomenon of chaotic scattering in continuous-time Hamiltonian systems in the presence of *weak dissipation*. Most previous works on classical chaotic scattering focused on purely conservative systems [1–4]. Note that conservative and dissipative systems admit a Hamilton function although the energy is not preserved in the dissipative cases [5] . A commonly studied setting is particle motion in a potential field consisting of a group of potential hills [2, 3]. In general, there exists a region where interactions between scattering particles and the potential occur, whereas outside the region, the potential is negligible so that the particle motions are essentially free. This region is often called the *scattering region* [1–4]. For many potential functions of physical interest, the corresponding classical Hamilton's equations of motion are nonlinear, rendering possible chaotic dynamics in the scattering region. Since the system is open, the region necessarily possesses "holes" for particles to enter and to escape. That is, particles from far away can enter the scattering region through one of the holes, experience chaotic dynamics in the region due to the interaction with the potential, and then exit the region through the same or a different hole. Because of the chaotic dynamics in the scattering region, particles with slightly different initial conditions (e.g., initial positions and momenta) can experience different paths of motion in the region and, consequently, they can spend drastically different times in the region and may exit through different holes in completely different directions. It is in this sense of sensitive dependence of the outcome of the scattering trajectory on the initial condition that the scattering is termed chaotic. In the past two decades or so, physical situations where chaotic scattering is relevant were identified, which include celestial mechanics [6], charged particle motions in electric and magnetic fields [7], hydrodynamical processes [8], atomic and nuclear physics [9], and solid-state semiconductor structures that are fundamental devices in nanoscience and nanotechnology [10].

For particles coming into the scattering region from far away, their lifetimes in the region must be finite. As a result, they exhibit chaotic dynamics but only for a finite amount of time, i.e., transient chaos [11, 12]. In this sense chaotic scattering can be regarded as a physical manifestation of transient chaos [1–4]. It is known in nonlinear dynamics that transient chaos is due to the existence of nonattracting chaotic invariant sets (chaotic saddles) in the phase space [2, 3, 13]. One way to physically see the presence of a chaotic saddle is through unstable periodic orbits. For instance, in the configuration of symmetric potential hills used to study various bifurcations to chaotic scattering [2, 3], there exist trajectories that bounce back and forth along the line segments connecting the centers of the hills. Insofar as the particles move exactly along these lines, they remain in the same paths, resulting in periodic motions which, in the phase space, correspond to periodic orbits. These orbits are unstable because an arbitrarily small deviation from the periodic paths can cause the particles to leave the paths and eventually leave the scattering region. To caracterize the scattering dynamics of the system we define "basin of attraction", as a set of initial conditions that leads to an attractor or fixed point. This term is associated with dissipative dynamical systems since an attractor is needed. In *open* conservative Hamiltonian systems we cannot talk about attractors or basins of attraction. For this case, we define "exit basin" in the same way that basin of attraction in dissipative systems, as a set of initial conditions that lead to a certain exit. In chaotic scattering, the boundaries separating the basins of different destinations are typically fractal sets [2, 14]. The main goal of this paper is to examine how weak dissipation may affect the basin structures.

In Hamiltonian systems, regular motions, i.e., motions on various Kol'mogorov-Arnol'd-Moser (KAM) tori [15], are also fundamental. Depending on whether there are KAM tori coexisting with chaotic saddles in the phase space, chaotic scattering may be characterized as either *hyperbolic* or *nonhyperbolic*. In hyperbolic chaotic scattering, all the periodic orbits are unstable and there are no KAM tori in the phase space. In this case, the particle decay law is exponential. To see this, consider an ensemble of initial particles randomly distributed in the scattering region. As time goes particles begin to escape from the region, so the number of particles in the region (or the survival probability of a particle) decreases with time. When chaotic saddles are the only dynamical invariant sets in the scattering region so that all periodic orbits are unstable, this decrease in the survival probability is necessarily exponential. In nonhyperbolic chaotic scattering, KAM tori coexist with chaotic saddles, which typically results in algebraic decay in the survival probability of a particle in the scattering region. Another goal of this paper is to address the effect of weak dissipation on particle decay law in nonhyperbolic scattering.

A recent work based on a class of two-dimensional, idealized discrete-time Hamiltonian maps examined the effect of dissipation on chaotic scattering in terms of the particle decay law and the fractal dimension of the chaotic saddle [17]. The finding was that for hyperbolic chaotic scattering, the exponential decay law remains unchanged in the presence of weak dissipation but, for nonhyperbolic chaotic scattering, the algebraic decay law is structurally unstable in the sense that it immediately becomes exponential in the presence of some amount of dissipation, no matter how small. This result is consistent with the fact that hyperbolic dynamics in Hamiltonian systems are typically structurally stable while nonhyperbolic dynamics are not. Here, we shall use a paradigmatic continuous-time model for chaotic scattering, the Hénon-Heiles system, and demonstrate that a similar result for the particle decay law and for the fractal dimension holds. More importantly, we will go beyond Ref. [17] to address the effect of weak dissipation on the basin topology of chaotic scattering. The surprising finding that we will report in this paper is that the Wada basin topology is *structurally stable* with respect to weak dissipations, even when the original Hamiltonian system exhibits nonhyperbolic chaotic scattering. To provide a solid basis for this result, we shall apply the mathematical conditions for Wada basins to our weakly dissipative Hénon-Heiles system.

In Sec. II, we describe the Hénon-Heiles system and discuss the basic scattering dynamics. In Sec. III, we estimate the fractal dimension for the conservative and dissipative cases. Section IV presents numerical results with the Wada basin topology and a mathematical argument. Conclusions and discussions are presented in Sec. V.

II. MODEL DESCRIPTION

The Hénon-Heiles system is described by the Hamiltonian

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3, \quad (1)$$

which defines the motion of a particle with unit mass in the two-dimensional potential

$$V(x, y) = \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3. \quad (2)$$

The system was originally proposed in 1964 to address the question of whether there exist more than two constants of motion in the dynamics of a galaxy model [23]. Since then it has become a paradigmatic model for studying nonlinear and chaotic dynamics in *continuous-time* Hamiltonian systems. For this potential we can distinguish two main types of motions, which correspond to bounded and unbounded orbits. According with the value of the energy we can say when the orbit is trapped in a region or escape from it. Specifically, this threshold value of the energy for which the particle can escape to the infinity is called *escape energy*, E_e. To calculate the escape energy E_e, it is necessary to find the value of the energy in the maxima of the potential, and this value is $E_e = 1/6 = 0.1666$. For values of energy above the threshold value the escapes are possible and the motions are unbounded, presenting the system three different exits.

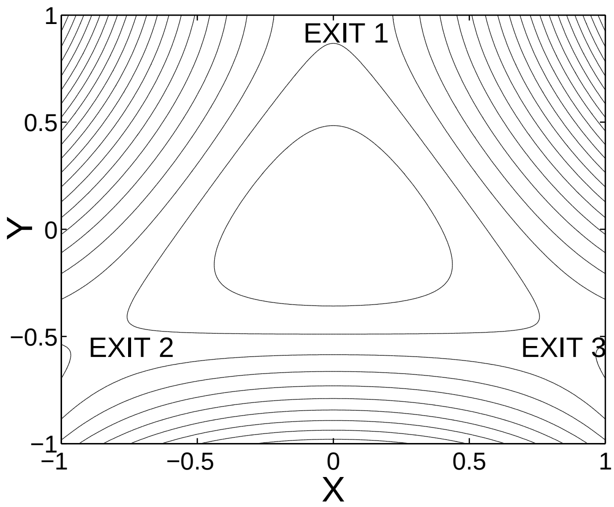


FIG. 1: Representative contours of the Hénon-Heiles potential. Closed curves correspond to energy $E < E_e = 1/6$. There are three symmetric destinations for scattering particles.

Figure 1 shows several contours of the potential as Eq. 2. There is a $2\pi/3$ rotational symmetry, with the center of the potential at the origin. For particles initiated from near the center with energy $E < E_e = 1/6$, they will be confined in the neighborhood of the center and thus generate bounded orbits in the phase space. Entering from outside into and escaping from the central region of the potential are possible only when the particle energy exceeds E_e. Since our interest is in scattering, we will focus on the $E > E_e$ regime. The triangular-like region around the center in Fig. 1 is thus the scattering region, the size of which depends on the particle energy. As indicated in Fig. 1, there are three symmetric channels for particles to exit the scattering region, giving rise to three qualitatively distinct scattering destinations. This allows Wada basin boundaries to occur.

For simulation convenience, we launch scattering particles from within the scattering region and examine their escaping trajectories. Specifically, the particles are distributed on a vertical line segment centered at $(x, y) = (0, 0)$ and they start their motions in different directions. That is, the subspace in the phase space from which scattering particles are initiated can be denoted by (y, θ), where θ is the angle of the initial velocity with respect to the x-axis. Figure 2 shows a typical trajectory with $E = 0.2$, where the particle spends a finite amount of time in the scattering region bouncing back and forth among the three potential peaks, before exiting through one of the escaping channels. A basic property of the Hénon-Heiles system is the existence of a class of highly unstable periodic orbits for $E > E_e$, called the *Lyapunov orbits*, which live near the

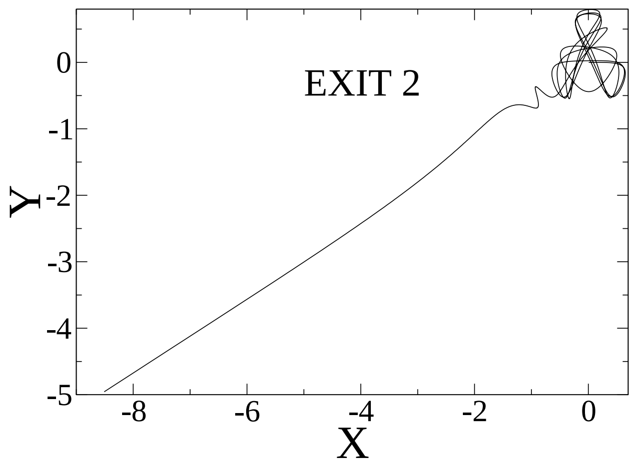


FIG. 2: A typical scattering trajectory in the Hénon-Heiles system with $E = 0.2$, where the particle escapes through exit 2 in Fig. 1.

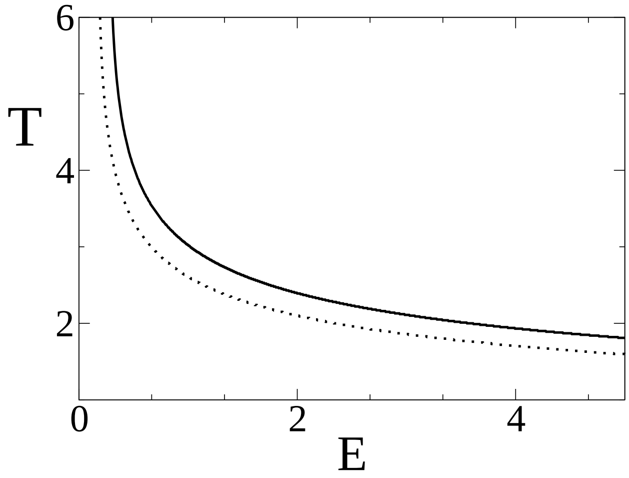
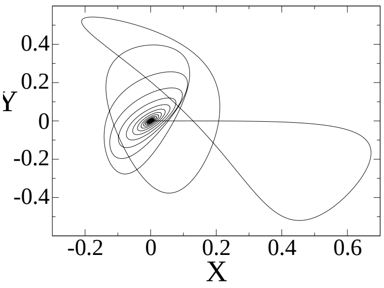


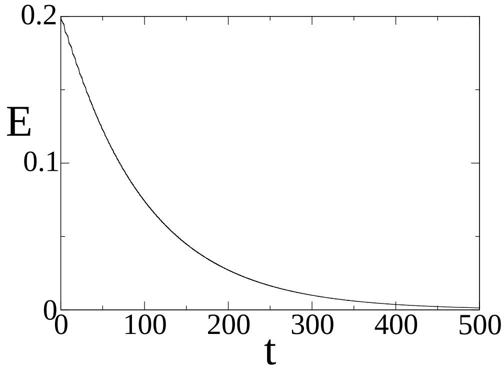
FIG. 3: Average delay time versus the energy. The time diverges as the particle energy approaches the threshold value E_e from above. (a) Dotted curve is for non-dissipative case. (b) Solid curve is for dissipative case with $\alpha = \beta = 10^{-3}$.

boundary of the scattering region. When a particle crosses a Lyapunov orbit in the outer direction, then it escapes to infinity and it never comes back. The Lyapunov orbits thus provide a meaningful criterion for measuring the delay times of particles in the scattering region even when the system is dissipative [24]. Apparently, the closer the particle energy is to E_e, the longer the delay time, and the time diverges as $E \to E_e$. This behavior is shown in Fig. 3 (dotted curve).

A physically meaningful way to introduce the dissipation is to add terms that are proportional



(a)



(b)

FIG. 4: (a) In the presence of dissipation ($\alpha = 0.1$ and $\beta = 0.1$), a particle permanently trapped in the scattering region. The corresponding trajectory approaches the fixed-point attractor in the phase space. (b) Exponential decay to zero of the particle energy when it approaches the attractor.

to the particle velocity in the Hamilton's equations of motion [25]. This results in the following model of *dissipative* Hénon-Heiles system:

$$\begin{aligned} \ddot{x} + x + 2xy + \alpha\dot{x} &= 0, \\ \ddot{y} + y + x^2 - y^2 + \beta\dot{y} &= 0, \end{aligned} \tag{3}$$

where α and β are dissipation parameters. Due to dissipation, relatively larger energies are required for scattering dynamics, as shown by the solid curve in Fig. 3 for $\alpha = \beta = 10^{-3}$. We see that, as the energy is decreased, the delay time tends to diverge at a larger value of the energy than that in the conservative case (dotted curve).

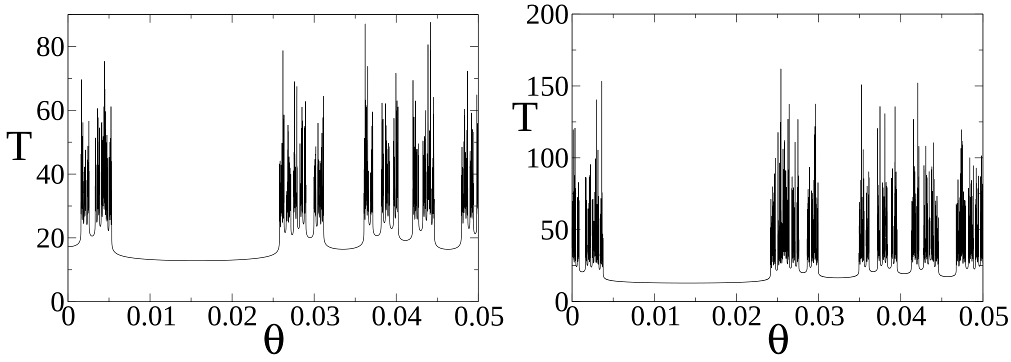
In the presence of dissipation, attractors can arise in the scattering region. For the *dissipative* Hénon-Heiles system, there is at least one such attractor located at the origin $(x, y) = (0, 0)$ with velocity $(\dot{x}, \dot{y}) = (0, 0)$, which clearly corresponds to a fixed-point attractor in the phase space. In this case, even for $E > E_e$, there is a probability that a particle can be trapped in the scattering region forever. Figure 4(a) shows such a trapped trajectory for $E = 0.2$, $\alpha = 0.1$, and $\beta = 0.1$, where it approaches asymptotically to the fixed-point attractor. During this process the particle energy decreases exponentially to zero, which is typical in dissipative dynamical systems, as shown in Fig. 4(b).

III. FRACTAL DIMENSION

An important result in nonhyperbolic chaotic scattering concerns the fractal dimension of the set of singularities in the scattering function. Lau *et al.* [26] argue, with numerical support, that the dimension is $D = 1$. This unity of the fractal dimension is a direct consequence of the algebraic-decay law associated with nonhyperbolic chaotic scattering, which can be seen intuitively by considering a zero-Lebesgue-measure Cantor set that has $D = 1$, through the construction explained in Ref. [26].

The scattering dynamics in the Hénon-Heiles system is typically nonhyperbolic, as KAM tori and chaotic saddles coexist in the phase space. Such a mixed phase-space structure will be shown in Sec. IV, but here we shall focus on the effect of dissipation on the fractal dimension. For nonhyperbolic chaotic scattering in the presence of dissipation, marginally stable periodic orbits in KAM islands can become stable attractors, turning their nearby phase-space regions into the corresponding basins of attraction [27]. This means that, part of the previous chaotic saddle now becomes part of the basins of the attractors. Most importantly, for the scattering dynamics, the converted subset supports orbits of the previous invariant set that are in the neighborhood of the KAM islands. These orbits are solely responsible for the nonhyperbolic character of the scattering, orbits which otherwise are scattered after a long, algebraic time. Due to the existence of dense orbits in the original chaotic saddle, the non-captured part of the invariant set remains in the boundaries of basins of the periodic attractors. Therefore the new invariant set is the asymptotic limit of the boundaries between scattered and *captured* orbits, rather than those between scattered and *scattered* orbits as in the conservative case. Chaos thus occurs on the nonattracting invariant set whose stable manifold becomes the boundary separating the basins of the attractors and of the scattering trajectories. Through this simple reasoning, we can see that the structure and the meaning of the Cantor set is fundamentally altered: in successive steps, a *constant* instead of a decreasing fraction in the middle of each interval is removed. As a result, the scattering dynamics becomes hyperbolic with exponential decay. The dimension of the Cantor set immediately decreases from unity as a dissipation parameter is turned on. This observation was verified numerically using a two-dimensional map [17]. We shall demonstrate here that the result holds for continuous-time systems as well.

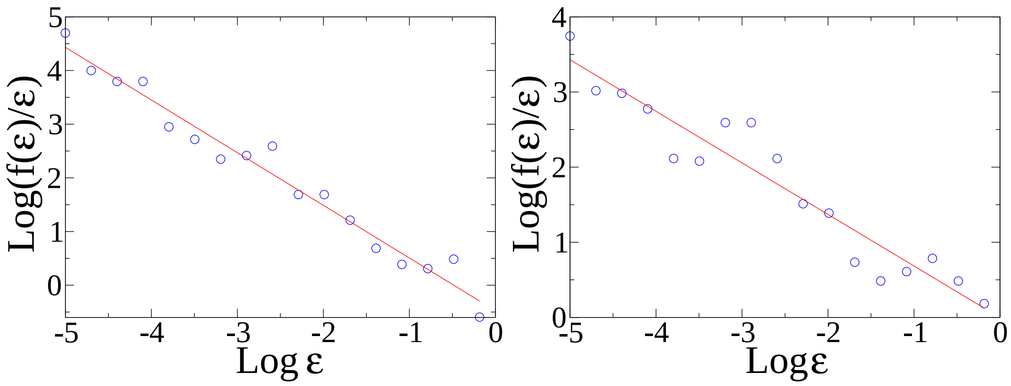
Figures 5(a) and 5(b) show, for $E = 0.19$ in the conservative and the dissipative ($\alpha = 10^{-4}$ and $\beta = 10^{-4}$) case, respectively, the delay-time function for scattering trajectories. To generate these



(a)

(b)

FIG. 5: Typical delay-time function for the conservative (a) and the weakly dissipative (b) Hénon-Heiles system with chaotic scattering ($E = 0.19$, $\alpha = 10^{-4}$, and $\beta = 10^{-4}$).



(a) Non-dissipative case

(b) Dissipative case

FIG. 6: For $E = 0.19$ in the Hénon-Heiles system so that there is chaotic scattering, algebraic scaling of $f(\epsilon)/\epsilon$ with ϵ. The absolute value of the slope from a linear fit gives the a good estimate for the fractal dimension of the set of singularities in the delay-time function. We obtain $D = 0.97 \pm 0.01$ for the conservative case and $D = 0.71 \pm 0.02$ for the weakly dissipative case.

figures, $n = 250$ particles are chosen at $y = 0$ with initial direction θ varying systematically from 0 to 0.05. We observe typical features of chaotic scattering in both cases: the functions contain both smooth parts and discontinuities and, in fact, they are singular on a fractal set. However, the fractal dimensions of the set of singularities in the two functions are markedly different, with the dimension value close to and less than unity in the conservative and dissipative case, respectively. To demonstrate this, we use the uncertainty algorithm [28] to numerically calculate the fractal dimension. In particular, for a fixed value of the "uncertainty" ϵ, we randomly choose an initial condition θ_0 and compute $\mid T(\theta_0) - T(\theta_0 + \epsilon) \mid$. If $\mid T(\theta_0) - T(\theta_0 + \epsilon) \mid > h$, where T is the delay time and h is a positive number, we say that θ_0 is uncertain with respect to ϵ. Otherwise θ_0 is certain. Many random initial conditions can be chosen, which yields $f(\epsilon)$, the fraction of the uncertain initial conditions. The quantity $f(\epsilon)/\epsilon$ typically scales with ϵ as

$$f(\epsilon)/\epsilon \sim \epsilon^{-D},$$

where D is the uncertainty dimension that is believed to have the same value as the box-counting dimension for typical dynamical systems [29]. Figures 6(a) and 6(b) show, for the Hénon-Heiles system in the conservative and the dissipative ($\alpha = 10^{-4}$ and $\beta = 10^{-4}$) case, respectively, $f(\epsilon)/\epsilon$ versus ϵ on a logarithmic scale, where the constant h is chosen (arbitrarily) to be $h = 0.01$. For Fig. 6(a), the estimated slope from a least-squares linear fit is $D = 0.97 \pm 0.01 \approx 1$. While for Fig. 6(b), the estimated slope is $D = 0.71 \pm 0.02 < 1$. Thus, the result that the fractal dimension decreases immediately from unity in the presence of weak dissipation, established previously exclusively for discrete-time maps, holds true for continuous-time Hamiltonian chaotic scattering systems as well. The variation of the fractal dimension D with the dissipation parameter $\mu = \alpha = \beta$ is shown in Fig. 7 for $E = 0.19$. We see that the dimension decreases rapidly from unity as μ is increased from zero. In fact, we expect the D-versus-μ curve to exhibit a cusp-like behavior for $\mu \gtrsim 0$, due to the metamorphic transition from algebraic to exponential decay in the survival probability of scattering particles caused by weak dissipation.

IV. WADA BASINS AND ITS PERSISTENCE UNDER WEAK DISSIPATION

In conservative Hamiltonian systems, like the Hénon-Heiles system, we cannot talk about attractors or basins of attraction. In this case, as we mentioned in the introduction, we talk about exit basins. The exit basins in the phase space have been obtained in Ref. [13], showing that the phase space has a very rich fractal structure as we can see in Fig. 8(a). As we have already mentioned in section II, if we introduce a small amount of dissipation in the system an attractor appears at the point of coordinates $(0, 0)$. As a consequence, now we can talk about basins of attraction. In particular, we have three exits and an attractor. Our phase space has now four different regions, three of them correspond to the three different exits and another one corresponds to the attractor. The boundaries of every region for a value of the energy close to the threshold value are fractals and it is very difficult to predict for close points the evolution of the system as it was

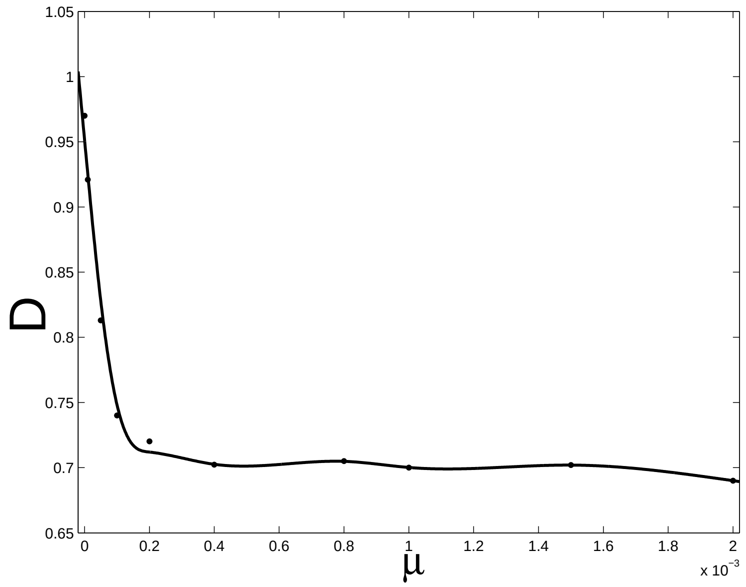
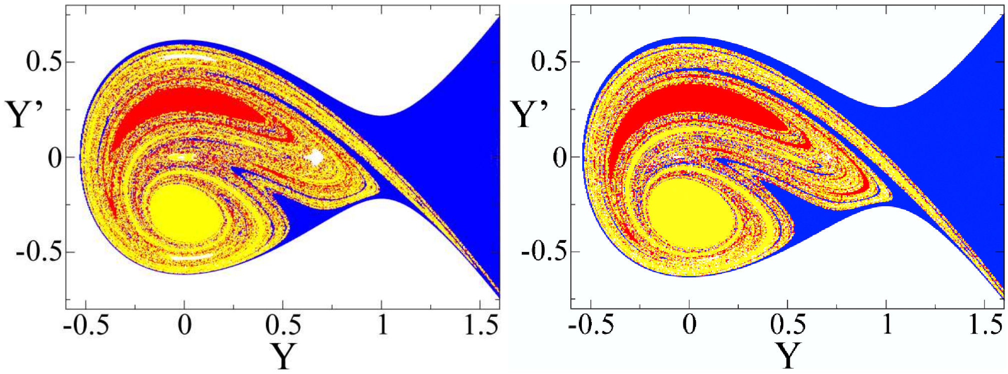


FIG. 7: The uncertainty dimension D versus the dissipation parameter $\mu = \alpha = \beta$. The rapid decrease in D for $\mu \gtrsim 0$ is characteristic of a cusp.

discussed in Ref. [13]. In Fig. 8(b) we can see the basins of attraction for a value of the energy above the threshold value $E > E_e$ and a value of dissipative parameters $\alpha = \beta = 10^{-4}$ where the system becomes hyperbolic, as a consequence to introduce dissipation, and the KAM islands are destroyed. We have seen from the preceding section that some aspects of nonhyperbolic chaotic scattering, e.g., the particle decay law and consequently the fractal-dimension characteristic, are structurally unstable with respect to weak dissipation. However, a surprising phenomenon is that the complex basin topology associated with chaotic scattering turns out to be persistent for both the Hamiltonian and the corresponding weakly dissipative system. To demonstrate this property numerically, we choose a two-dimensional plane in the three-dimensional phase space and launch a large number of scattering particles from this plane. The locations of the initial particles can be distinguished by examining through which escaping channels they leave the scattering region. Figure 8(a) shows, for the Hénon-Heiles system in the absence of dissipation, such distinct sets of initial conditions in the plane $(y, \dot{y})$, where the particle energy is set to be $E = 0.19$ and the initial x-coordinate of the particles is $x(0) = 0$. To generate Fig. 8(a), a uniform grid of 500×500 initial conditions was chosen in the region $(-1 \leq y \leq 2, -1 \leq \dot{y} \leq 1)$. In Fig. 8(a), the set of blue, red and yellow dots denote initial conditions resulting in trajectories that escape through channel 1, 2 and 3 (Fig. 1), respectively, and the white regions inside the plotted structure denote the KAM



(a) (b)

FIG. 8: For $E = 0.19$ in the Hénon-Heiles system, (a) the basins of scattering particles in the conservative case. In this case, there are three destinations, and initial conditions going to the destinations are distinguished by three colors and the white regions inside the plotted structure denote the KAM islands. (b) The basin structure in the presence of a small amount of dissipation ($\alpha = 10^{-4}$ and $\beta = 10^{-4}$). Due to the dissipation, an additional destination arises: the fixed-point at the center of the scattering region. Four colors are then needed to distinguish the initial conditions. See text for simulation details.

islands. We see a complex, fractal-like basin structure. In fact, it can be shown that the basins are not only fractals, but are also Wada (to be precisely defined below) [13]. Figure 8(b) shows, for the same simulation setting but with weak dissipation ($\alpha = 10^{-4}$ and $\beta = 10^{-4}$), the exit basins. Due to the appearance of the fixed-point attractor at the center of the scattering region, now four colors are needed to distinguish the initial conditions according to the four possible destinations: exits 1-3 and the attractor. In particular, the colors $blue$, red, and $yellow$, denote initial conditions that escape through exits 1-3, respectively, and white regions inside the structure plotted denote the basin of the fixed-point attractor. Qualitatively, we observe a similar mixture of basins as in the conservative case, suggesting that the Wada property persists under weak dissipation. Figures 9(a-d) show, for $E = 0.19$, the basins for $\mu = 5 \times 10^{-4}, 10^{-3}, 10^{-2}$, and 10^{-1}, respectively, the basin structures. Apparently, as the dissipation parameter is increased, the structures appear "less fractal", as suggested by Fig. 7.

We now argue that the basins seen in Fig. 8(b) possess the Wada property. In a nonlinear dynamical system, situation can arise where the set of boundary points common to more than two basins of attraction is fractal. Mathematically, a basin is Wada if any boundary point also belongs to the boundaries of at least two other basins [18–20], i.e., every open neighborhood

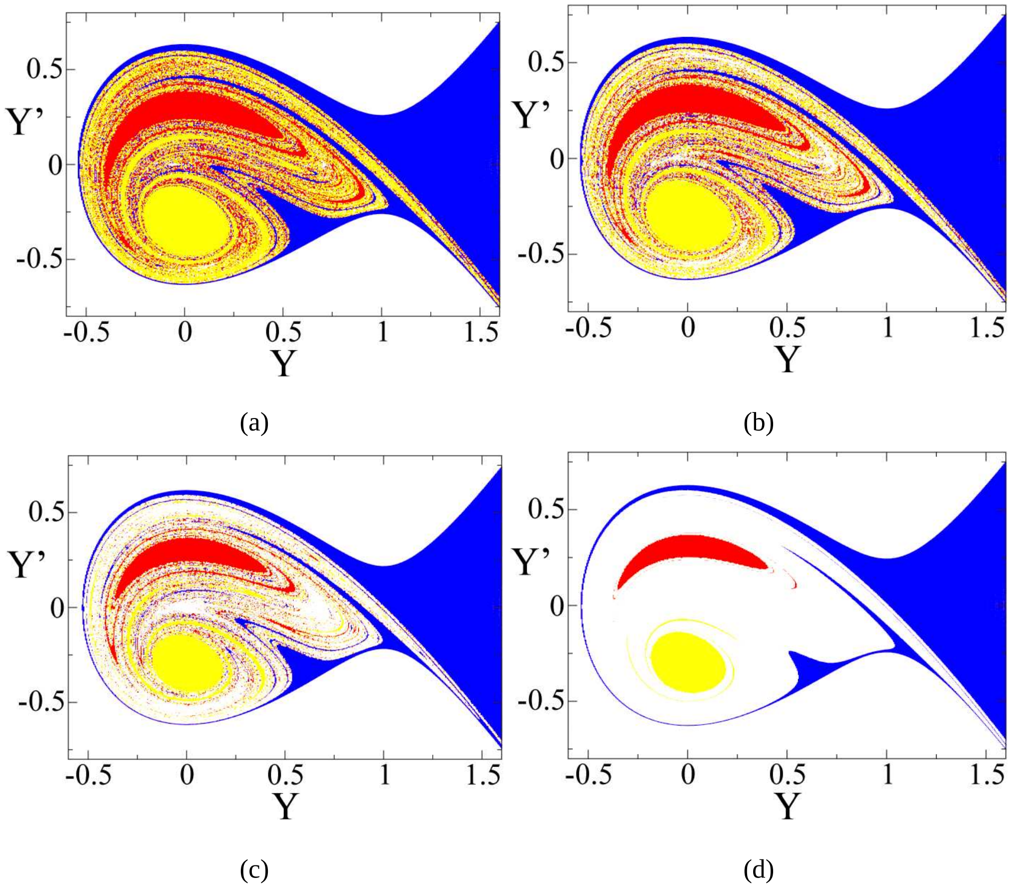


FIG. 9: (a-d) Basins of scattering destinations and of the fixed-point attractor at the center of the scattering region for $E = 0.19$ and for $\mu = 5 \times 10^{-4}, 10^{-3}, 10^{-2}$, and 10^{-1}, respectively.

of a point belonging to a Wada basin boundary has a nonempty intersection with at least three different basins. If a dynamical system possesses Wada basins, the degree of unpredictability of destinations can be more severe than the case where there are fractal basin boundaries with only two destinations [13, 14, 21]. Wada basin boundaries in chaotic scattering have been recently observed experimentally with a simple optical system [22]. The common occurrence of Wada basin boundaries in nonlinear dynamical systems was first pointed out by Kennedy and Yorke in 1991 [18].

For two-dimensional invertible maps or equivalently, three-dimensional flows, the mechanism for Wada basin boundaries is well understood, thanks to the rigorous mathematical work by Kennedy, Nusse, and Yorke [18, 20]. In particular, Kennedy and Yorke proved a theorem [18] which states that, if $\mathbf{p}$ is a periodic point on the basin boundary, if the following two conditions are satisfied: (1) its unstable manifold intersects every basin (**Main Condition**), and (2a) its stable manifold is dense in each of the basin boundaries or (2b) this is the only periodic point accesible from the basin of interest, then the basins have the Wada property. The secondary condition (2a) can be intuitively understood by referring to Fig. 10, where there are a number of K coexisting basins denoted by B_1, B_2, ..., B_K. Suppose $\mathbf{p}$ is a periodic point on the boundary of B_1, which is accessible to B_1. Let $W^s(\mathbf{p})$ and $W^u(\mathbf{p})$ be the stable and the unstable manifold of $\mathbf{p}$ (note that $W^s(\mathbf{p})$ is the basin boundary of B_1). Now arbitrarily choose a point $\mathbf{x} \in W^s(\mathbf{p})$ and imagine a circle $C_\epsilon(\mathbf{x})$ of radius ϵ centered at $\mathbf{x}$. Since $W^u(\mathbf{p})$ intersects every basin, $C_\epsilon(\mathbf{x})$ must contain points of every basin, which can be seen by considering a one-dimensional curve segment D_k in the basin B_k, which intersects $W^u(\mathbf{p})$, for $k = 1, \ldots, K$. Under inverse iterations of the map, the images of the curves will be arbitrarily close to the stable manifold of $\mathbf{p}$ and therefore be in $C_\epsilon(\mathbf{x})$. In fact, this is guaranteed mathematically by the λ-lemma due to Palis [31] which states that there exists a positive integer n such that $(\mathbf{F}^{-1})^{(n)}(D_k) \cap C_\epsilon(\mathbf{x})$ is nonempty. We thus see that the boundary of B_1 must be the boundaries of all other basins. Since $W^s(\mathbf{p})$ is dense in each of the basin boundaries, all boundaries must be common to all basins and hence the Wada property. For our especific case, the Hénon-Heiles, the secondary condition (2b) has been shown in Ref. [13].

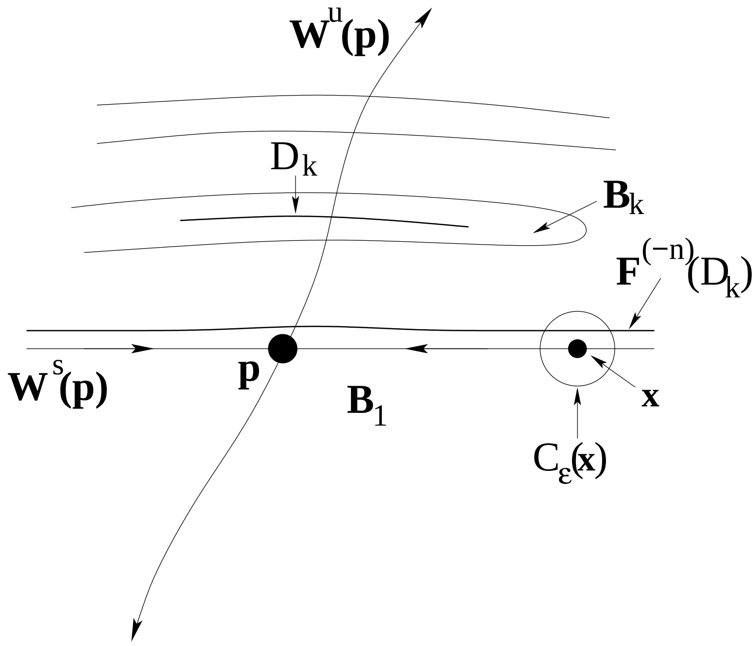
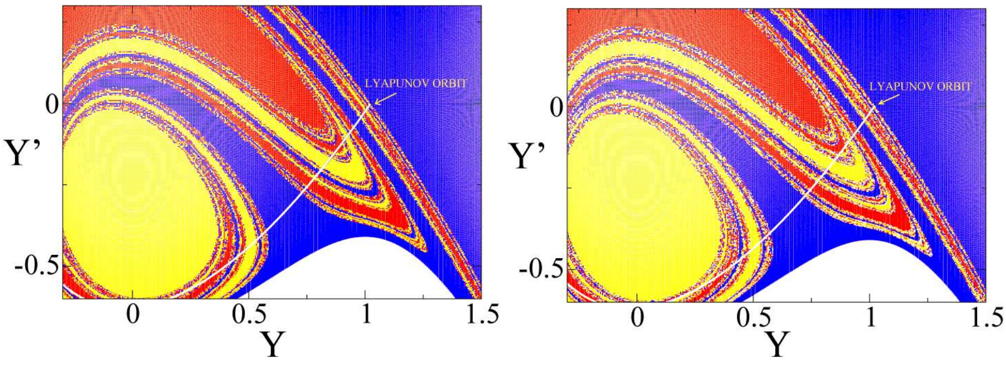


FIG. 10: Schematic illustration of the Kennedy-Yorke theorem establishing the Wada property. See text for details.

Computationally, to verify condition (1), one can plot a piece of the unstable manifold, trace it under the dynamics, and determine whether it intersects all basins. This is feasible for our weakly



(a) Hamiltonian case (b) Dissipative case

FIG. 11: For $E = 0.25$ in the Hénon-Heiles system, a segment of the unstable manifold of one of the three Lyapunov orbits, an unstable periodic orbit accessible to at least one basin. That the segment intersects all four basins suggests the Wada property of the basins, (a) in the Hamiltonian case, (b) in the dissipative case with $\alpha = 10^{-4}$, and $\beta = 10^{-4}$.

dissipative Hénon-Heiles system, because all destinations for scattering trajectories are known. (In situations where one cannot be certain if all basins have been found, the technique of basin cells [20] can be used to determine *rigorously* whether the basins are Wada.) To do so, we first locate one of the Lyapunov orbits, an unstable periodic orbit [13] accessible to at least one basin of attraction [32], and compute its unstable manifold by evolving a large number of initial conditions chosen in a small neighborhood of the orbit forward in time. Note that , as we mentioned in section II, the Lyapunov orbits exit for the dissipative case. Moreover, if they are periodic orbits for the conservative case then they will continue being periodic when dissipation is introduced in the system[24]. Figure 11 shows, for $E = 0.25$, $\alpha = 10^{-4}$, and $\beta = 10^{-4}$, a segment of the unstable manifold of a Lyapunov orbit. We observe that, indeed, the unstable manifold intersects all four basins, suggesting the Wada property.

V. CONCLUSIONS AND DISCUSSIONS

While chaotic scattering has been studied for more than two decades [1–4], almost all works focused exclusively on conservative Hamiltonian systems with no dissipation. Indeed, strictly Hamiltonian systems are fundamental to a great many physical problems, especially those in celestial mechanics, atomic and nuclear physics. This being true, there are also physical situations where dissipation exists, such as particle advection in fluids. It is thus important to address how weak dissipation affects the many known characteristics of chaotic scattering. In this regard, a recent work [17] based on a class of two-dimensional maps established that weak dissipation can affect nonhyperbolic chaotic scattering in a drastic way: the associated properties of algebraic-decay law and unity of the fractal dimension are immediately destroyed by weak dissipation. Interestingly, we discovered that the Wada basin topology remains qualitatively unchanged for nonhyperbolic chaotic scattering when weak dissipation is present. Thus, one can expect the Wada basin topology to be more common than, say, the algebraic decay of particles in nonhyperbolic chaotic scattering.

That dissipation can be important for Hamiltonian systems can be seen through the following example in fluid mechanics. It is known that the advective dynamics of idealized particles in two-dimensional, incompressible flows can be described as Hamiltonian [33]. For instance, consider such a flow characterized by a stream function $\Psi(x, y, t)$. For a particle with zero inertia and zero size, its trajectory in the flow obeys the following equations: $dx/dt = \partial\Psi(x, y, t)/\partial y$ and $dy/dt = -\partial\Psi(x, y, t)/\partial x$, which are the standard Hamilton's equations of motion generated by the Hamiltonian $H(x, y, t) = \Psi(x, y, t)$. That is, the particle velocity $\mathbf{v}(x, y, t) = (dx/dt, dy/dt)$ follows exactly the flow velocity $\mathbf{u}(x, y, t)$, as given by the right-hand side of the equations. This idealized picture changes completely when particles have finite inertia and size. In this realistic case, the particle velocity is generally not the same as the flow velocity and the equations of motion are no longer Hamilton's equations. The resulting dynamical system is no longer Hamiltonian but dissipative instead [34]. Considering that in an open Hamiltonian flow, ideal particles coming from the upper stream must necessarily go out of the region of interest in finite time, the formation of attractors of inertial particles is remarkable. Suppose these physical particles are biologically or chemically active. That they can be trapped permanently in some region in the physical space is of great interest or concern. Our work suggests that Wada basin boundaries may be a common feature when inertial particles can go to several distinct destinations.

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