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A NEW MECHANISM OF THE CHAOS SUPPRESSION

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Abstract. The standard Melnikov method for analyzing the onset of chaos in the vicinity of a separatrix is used to explore the possibility of suppression of chaos of a certain class of dynamical systems. For a given dynamical system we apply an external perturbation, which we call the stabilizing perturbation, with the goal that after its action the chaos present in the system is suppressed. We apply this method to the nonlinear pendulum as a paradigm, and obtain some analytical expressions for the corresponding external perturbations that eliminate chaotic behavior. Numerical simulations in the pendulum show a complete agreement with the analytical results.

1. **Introduction.** Most nonlinear dynamical systems may possess chaotic behavior for a certain choice of parameters. Since there are situations for which this behavior might be undesirable, different methods have been developed in the past years to suppress or control chaos. The idea that chaos may be suppressed goes back to the publications [1, 2] where it has been proposed to perturb periodically the system parameters with the final effect of suppression of chaos. Later this idea has been analytically verified [3]. The method of controlling chaos has been introduced in the paper [4] (the history of this question see in review [5]).

There are some studies of this problem using mainly numerical simulations, although analytical methods have also been used. One of them, the Melnikov method [6], is a sufficiently effective tool for the analysis of chaotic systems. Even though the method is only approximate, due to its perturbative nature, it provides a way to obtain a general expression relating different parameters of the system and reveals a threshold value for the onset of chaos.

Different other methods have been applied for the suppression of chaos (see, e.g. [7–10] and refs. cited therein). In these papers, Jacobian Elliptic Functions instead of the common trigonometric functions as the perturbations acting on the dynamical system have been basically used.

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The main idea of this paper is to apply a general perturbation to the dynamical system in such a way that after application of the Melnikov analysis, the dynamical system finally shows no chaotic behavior. In other words, given a certain dynamical system for which chaos exist for a given choice of parameters, the challenge is to find an appropriate perturbation, that we call the *function of stabilization*, which would convert the dynamical system into non-chaotic. Some preliminary results concerning this idea has been partially described in [11].

This is done analytically for a general two-dimensional system, and then the results are applied to the nonlinear pendulum for a choice of parameters for which it shows chaotic behavior. Afterwards, we show by using numerical computations that there is a complete agreement with the analytical results.

The physical meaning [12] of the stabilizing perturbation in the case of the pendulum corresponds to a series of hits acting on the pendulum. Nevertheless, what we show here is that we can find this stabilizing perturbation, which eventually will depend on each dynamical system. The idea can be useful for a general class of dynamical systems for which we can try to find the appropriate means of avoiding the appearance of chaos. The same idea could be useful for dissipative systems and for conservative systems as well.

2. **Stabilization of chaotic motion.**

In this section we address the problem that we have raised in the introduction, that is, we apply the Melnikov method, which gives a criterion of the chaos appearance, to the analysis of the system behavior under external perturbations. The idea is that such an approach can give us an analytical expression of the perturbations which leads to the chaos suppression phenomenon.

We explain the idea by using a general two-dimensional dynamical system subjected to a time-periodic external perturbation, and consequently possessing a three dimensional phase space, and our results are then applied to the nonlinear pendulum system.

2.1. **Melnikov function.**

It is well known, that in Hamiltonian systems separatrices can split. In this case stable and unstable manifolds of a hyperbolic point do not coincide, but intersect each other in an infinite number of homoclinical points (usually the motion in the $(n+1)$–dimensional phase space $(x_1, \cdots, x_n, t)$ is considered in the projection onto a n–dimensional hypersurface $t = \text{const}$ (Poincaré section)). The presence of such points gives us a criterion for the observation of chaos. This criterion can conveniently be obtained by the *Melnikov function* (MF), which "measures" (in the first order of a small perturbation parameter) the distance between stable and unstable manifolds.

Melnikov analysis is based on the paper [6]. First, we consider a two-dimensional dynamical system under the action of a periodical perturbation with the property of having a unique saddle point:

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \varepsilon \mathbf{f}_1(\mathbf{x}, t), \tag{1}$$

Let furthermore $\mathbf{x}_0$ be the separatrix of the unperturbed system $\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x})$. Then the MF at any given time t_0 is defined as follows:

$$D(t_0) = -\int_{-\infty}^{+\infty} \mathbf{f}_0 \wedge \mathbf{f}_1 \Big|_{\mathbf{x}=\mathbf{x}_0(t-t_0)} dt,$$

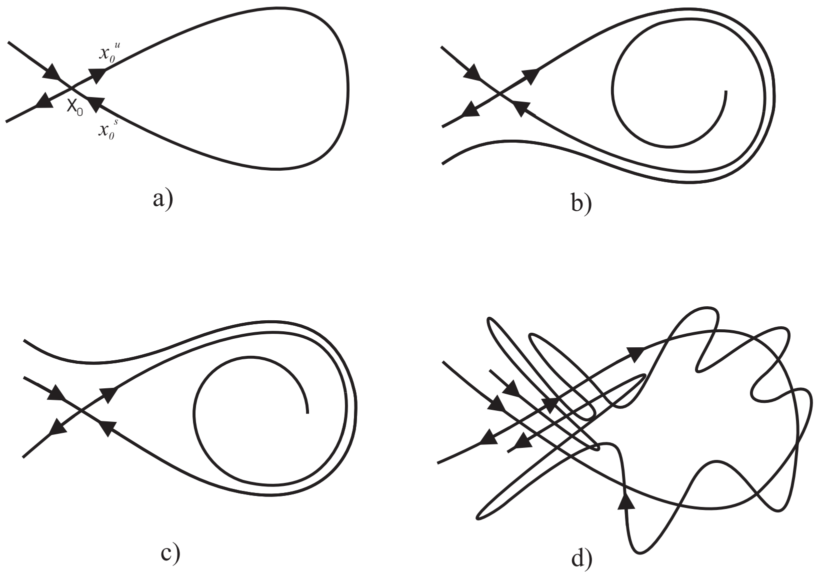

FIGURE 1. Poincaré section t = const (mod T) of the system (Eq. 1) for $\varepsilon = 0$ (Fig. 1a) and $\varepsilon \neq 0$ (Figs. 1b,1c,1d).

where the integral is taken along the unperturbed separatrix $\mathbf{x}_0(t - t_0)$ and the integrand is $\mathbf{f}_0 \wedge \mathbf{f}_1 = f_{0x}f_{1y} - f_{0y}f_{1x}$.

In general, in dissipative systems one can observe three possibilities for the MF: either $D(t_0) < 0$ (Fig. 1b), $D(t_0) > 0$ (Fig. 1c) for any t_0 or $D(t_0)$ changes its sign for some t_0 (Fig. 1d). Only in the last case chaotic dynamics arises. Thus, the MF determines the character of the motion near the separatrix. Note that the Melnikov method has a perturbative (to first order) character, thus, its application is allowed only for trajectories which are sufficiently close to the unperturbed separatrix. Moreover this method is valid only for systems with $\varepsilon \ll 1$.

2.2. **Function of stabilization.** The Melnikov method has been applied in a lot of typical physical situations (see Refs. [7, 13–17]) in which homoclinic bifurcations occur. Here we consider an application of the Melnikov method to the analysis of the chaos suppression phenomenon in systems with separatrix loops. Such an approach allows us to find an analytical expression of the perturbations for which the Melnikov distance $D(t_0)$ does not change sign (see also [18]) suppressing the chaotic behavior and stabilizing the orbits of the system.

We consider the problem of stabilization of chaotic behavior in systems with separatrix contours that can be described by equation (1)

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \varepsilon \mathbf{f}_1(\mathbf{x}, t),$$

where $\mathbf{f}_0(\mathbf{x}) = (f_{01}(\mathbf{x}), f_{02}(\mathbf{x}))$, $\mathbf{f}_1(\mathbf{x}, t) = (f_{11}(\mathbf{x}, t), f_{21}(\mathbf{x}, t))$. For this equation the Melnikov distance $D(t_0)$ is given by $D(t_0) = -\int\limits_{-\infty}^{\infty} \mathbf{f}_0 \wedge \mathbf{f}_1 dt \equiv I[g(t_0)]$. Let us assume that $D(t_0)$ changes its sign. To suppress chaos we should get *a function of stabilization* $\mathbf{f}^*(\omega,t)$ that leads us to a situation when separatrices are not intersected:

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \varepsilon\left[\mathbf{f}_1(\mathbf{x},t) + \mathbf{f}^*(\omega,t)\right], \tag{2}$$

where $\mathbf{f}^*(\omega,t) = (f_1^*(\omega,t), f_2^*(\omega,t))$. Suppose $D(t_0) \in [s_1,s_2]$ and $s_1 < 0 < s_2$. After the stabilizing perturbation $\mathbf{f}^*(\omega,t)$ is applied we have two cases: $D^*(t_0) > s_2$ or $D^*(t_0) < s_1$, where $D^*(t_0)$ — Melnikov distance for system (2). We consider the first case (analysis for the second one is similar). Then

$$I[g(t_0)] + I[g^*(\omega,t_0)] > s_2, \tag{3}$$

where $I[g^*(\omega,t_0)] = -\int\limits_{-\infty}^{+\infty} \mathbf{f}_0 \wedge \mathbf{f}^* dt$. Expression (3) is true for all left hand side values of inequality that is greater than s_2. It is derived that $I[g(t_0)] + I[g^*(\omega,t_0)] = s_2 + \chi = const$, where $\chi, s_2 \in \mathbb{R}^+$. Therefore $I[g^*(\omega,t_0)] = \mathrm{const} - I[g(t_0)]$. On the other hand, $I[g^*(\omega,t_0))] = -\int\limits_{-\infty}^{\infty} \mathbf{f}_0 \wedge \mathbf{f}^* dt$. We choose $\mathbf{f}^*(\omega,t)$ from the class of functions that are absolutely integrable on an infinite interval such that they can be represented in Fourier integral form. Then $\mathbf{f}^*(\omega,t) = \mathrm{Re}\{\hat{A}(t)e^{-i\omega t}\}$. Here we suppose that $\hat{A}(t) = (A(t), A(t))$ i.e., assume that the regularizing perturbations applied to both components of the equation (2) are identical. Therefore $-\int\limits_{-\infty}^{\infty} \mathbf{f}_0 \wedge \left\{\hat{A}(t)e^{-i\omega t}\right\} dt = \mathrm{const} - I[g(t_0)]$. The inverse Fourier transform yields: $\mathbf{f}_0 \wedge \hat{A}(t) = \int\limits_{-\infty}^{\infty} \left(I\left[g\left(t_0\right)\right] - \mathrm{const}\right) e^{i\omega t} d\omega$. Hence,

$$A(t) = \frac{1}{f_{01}(x) - f_{02}(x)} \int\limits_{-\infty}^{\infty} \left(I\left[g\left(t_0\right)\right] - \mathrm{const}\right) e^{i\omega t} d\omega. \tag{4}$$

Here $A(t)$ can be interpreted as the amplitude of the "stabilizing" perturbation. Thus, for system (1) the external stabilizing perturbation has the form:

$$f^*(\omega,t) = \mathrm{Re}\left[\frac{e^{-i\omega t}}{f_{01}(x) - f_{02}(x)} \int\limits_{-\infty}^{\infty} \left(I\left[g\left(t_0\right)\right] - \mathrm{const}\right) e^{i\omega t} d\omega\right]. \tag{5}$$

Let us now consider the stabilization problem for systems of the next type

$$\begin{aligned} \dot{x} &= P(x,y), \\ \dot{y} &= Q(x,y) + \varepsilon[f(\omega,t) + \alpha F(x,y)], \end{aligned} \tag{6}$$

where $f(\omega,t)$ is a time periodic perturbation, $P(x,y)$, $Q(x,y)$, $F(x,y)$ are some smooth functions and α is the dissipation.

We investigate the case which is typical for applications with a single hyperbolic point in the origin $x = y = 0$ when $P(x,y) = y$. Let $x_0(t)$ be solution on the separatrix. In the presence of the perturbation the Melnikov distance $D(t_0)$ for the system (6) may be written as

$$D(t_0) = -\int_{-\infty}^{\infty} y_0(t-t_0)\,[f(\omega,t) + \alpha F(x_0,y_0)]\,dt \equiv I[g(\omega,\alpha)], \tag{7}$$

where $y_0\,(t) = \dot{x}_0\,(t)$. Let us suppose again that the Melnikov function (7) changes sign, i.e., separatrices are intersected. We will find an external regularizing perturbation $\mathbf{f}^*(\omega,t) = \mathrm{Re}\{\hat{A}(t)e^{-i\omega t}\}$ that stabilizes the system dynamics:

$$\begin{aligned} &\dot{x} = y, \\ &\dot{y} = Q(x,y) + \varepsilon[f(\omega,t) + \alpha F(x,y) + f^*\,(\omega,t)]. \end{aligned} \tag{8}$$

It is significant to note that since the system (6) depends on parameter α then such stabilization should be made at every fixed value of this parameter and further, instead of $I[g(\omega,\alpha)]$, we will write $I[g(\omega)]$. For (8) we have $f_{01} = y$, $f_{02} = Q(x,y)$ and $\hat{A}(t) = (0, A(t))$. Consequently the value $A\,(t)$ has a form

$$A\,(t) = \frac{1}{y_0\,(t-t_0)} \int_{-\infty}^{\infty} (I\,[g\,(\omega)] - \mathrm{const})\,e^{i\omega t} d\omega. \tag{9}$$

So, for (8) the stabilizing function can be represented as

$$f^*\,(\omega,t) = \mathrm{Re}\left[\frac{e^{-i\omega t}}{y_0\,(t-t_0)} \int_{-\infty}^{\infty} (I\,[g\,(\omega)] - \mathrm{const})\,e^{i\omega t} d\omega\right]. \tag{10}$$

Now, let us find a regularizing perturbation in the case when the Melnikov function $D(t,t_0)$ admits an *additive* shift from its critical value.

Again, we analyze the case when $D^*(t_0) > s_2$ is satisfied. Suppose that α_c corresponds to the critical value of the Melnikov function, $I_c = I[g(\omega, \alpha|_{\alpha=\alpha_c})]$. Then, a subcritical Melnikov distance can be expressed as $I_{out} = I_c - a$, where $a \in \mathbb{R}^+$ is constant. Assuming that the system perturbed by $f^*(\omega,t)$ exhibits regular behavior, we have

$$I' = I_{out} + I\,[g^*\,(\omega)] > s_2. \tag{11}$$

Here $I\,[g^*\,(\omega)] = -\int_{-\infty}^{+\infty} y_0\,(t-t_0)\,f^*\,(\omega,t)\,dt$. On the other hand, it is obvious that we can take any I' a fortiori greater than I_c:

$$I' = I_c + a > s_2. \tag{12}$$

Now, equating the left-hand sides of (11) and (12), we obtain $I\,[g^*\,(\omega)] = 2a$. Substituting $f^*\,(\omega,t) = \mathrm{Re}\{A(t)e^{i\omega t}\}$ into the expression for $I[g^*(\omega)]$, we find $-\int_{-\infty}^{\infty} e^{i\omega t} A(t) y_0(t-t_0) dt = 2a$. The inverse Fourier transform yields $A\,(t)\,y_0\,(t-t_0) = -2a \int_{-\infty}^{\infty} e^{-i\omega t} d\omega$. Hence,

$$A\,(t) = -\frac{2a}{y_0(t-t_0)} \int_{-\infty}^{\infty} e^{-i\omega t} d\omega = -\frac{4\pi a \delta(t)}{y_0(t-t_0)}. \tag{13}$$

Thus, the dynamics of the systems that admit an additive shift from the critical value of the Melnikov function $D(t_0)$ are regularized by the perturbation:

$$f^*(\omega, t) = -\frac{4\pi a\delta(t)}{y_0(t-t_0)}\cos(\omega t), \tag{14}$$

where $\delta(t)$ is a Dirac delta–function defined as follows:

$$\delta(t) = \begin{cases} 0, & t \neq 0, \\ \infty, & t = 0. \end{cases}$$

In the general case, if $f_0 = (f_{01}(x), f_{02}(x))$, then we obviously obtain

$$f^*(\omega, t) = -\frac{4\pi a\delta(t)}{f_{01}(x) - f_{02}(x)}\cos(\omega t). \tag{15}$$

From the physical point of view, the obtained results mean that the dynamics of the chaotic system are stabilized by a series of "kicks".

3. **Nonlinear pendulum.** We apply the above analysis to the nonlinear pendulum [19]. The equation of the pendulum with dissipation and an external periodic perturbation is written as

$$\ddot{x} + \sin x = \varepsilon\left[\gamma\cos\omega t - \alpha\dot{x}\right]. \tag{16}$$

The unperturbed Hamiltonian of this system has the form: $H_0 = \dot{x}^2/2 - \cos x$. Phase space of the pendulum is 2π–periodical via x with hyperbolic saddle points in $(\pm\pi, 0)$ and a center in $(0, 0)$. The present system has three kind of solutions: oscillations, rotations and the separatrix motion. Oscillations are bounded motions and correspond to the case for which $|x(t)| < \pi$. Rotations are unbounded and correspond to either $y(t) < 0$ or $y(t) > 0$, where $y(t) = \dot{x}(t)$. And finally separatrix motion corresponds to the motion between fixed points, forming a heteroclinic orbit in phase space. We are interested in the last type of solutions which has the form:

$$\begin{aligned} x_0(t) &= 4\arctan\exp(\pm t) - \pi, \\ y_0(t) &= \dot{x}(t) = \pm 2/\cosh t, \end{aligned} \tag{17}$$

where the signs refer to the upper and lower half planes. The Melnikov distance that corresponds to the equation (16) was computed in [17]

$$D^{\pm}(t_0, \omega) = -\alpha\int_{-\infty}^{+\infty}(y_0(t))^2dt \pm \gamma\cos\omega t_0\int_{-\infty}^{+\infty}\sin(x_0(t))y_0(t)\cos\omega t dt. \tag{18}$$

After computing the integrand we obtain [17]:

$$D^{\pm}(t_0, \omega) = -4\alpha B\left(\frac{1}{2}, 1\right) \pm \frac{2\pi\gamma}{\cosh\left(\frac{\pi\omega}{2}\right)}\cos\omega t_0, \tag{19}$$

where $B(r, s)$ is the Euler β function.

It is easy to see that in this case the Melnikov function $D^{\pm}(t_0, \omega)$ obviously admits an additive shift from its critical values. Therefore, for the case of a chaotic nonlinear oscillator an external function of stabilization can be chosen in the following way

$$f^*(\omega, t) = -\frac{4\pi a\delta(t)}{y_0(t)}\cos\omega t, \tag{20}$$

where $y_0(t)$ is the solution on the unperturbed separatrix. We apply this stabilizing perturbation to the upper and lower branches of heteroclinical contour.

From the physical point of view, the obtained results mean that dynamics of the pendulum are regularized by series of "kicks".

4. **Application to the pendulum.** We have performed computer simulations on the nonlinear pendulum equation (16) with the following parameters: $\alpha = 0.1$, $\gamma = 6$, $\omega = 0.5$, where we took $\varepsilon = 0.1$ and we have used the value $a = 0.082$ in order to compute the stabilization function. It is known that these values correspond to the chaotic motion in the system. The phase portrait and relevant time evolution are shown in Fig. 2.

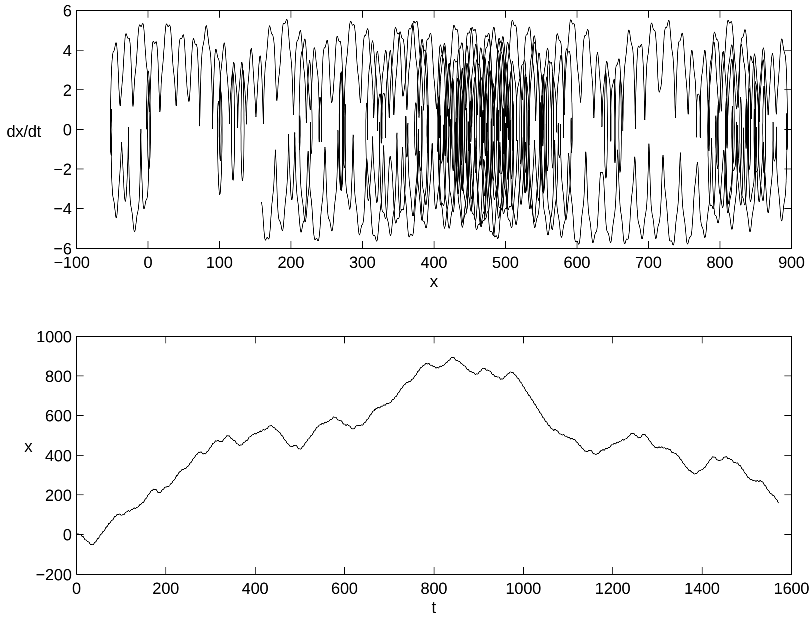

FIGURE 2. The figure shows a chaotic orbit in phase space and the chaotic shape of the time evolution of $x(t)$ for the pendulum $\ddot{x} + \sin x = \varepsilon\,[\gamma \cos \omega t - \alpha \dot{x}]$, where the parameters take the values $\alpha = 0.1$, $\gamma = 6$, $\omega = 0.5$ and $\varepsilon = 0.1$.

After introduction of the stabilizing perturbation $f^*(\omega, t)$ (see Eq. 20), the corresponding phase portrait and the solution $x(t)$ of the nonlinear oscillator equation is shown in Fig. 3. Another method of detecting the chaos suppression phenomenon is to consider the spectral properties of the solution $x(t)$ of the nonlinear pendulum equation. By the standard Fourier transform we can get the power spectrum. In Fig. 4 power spectra of the solution $x(t)$ for the pendulum equation (16) corresponding to Fig. 2 and Fig. 3, are shown. One can clearly see that by means of the stabilizing perturbation (see Eq. 20) it is possible to suppress chaos.

Thus, the obtained numerical analysis are in a good agreement with the analytical results described in §2 and §3.

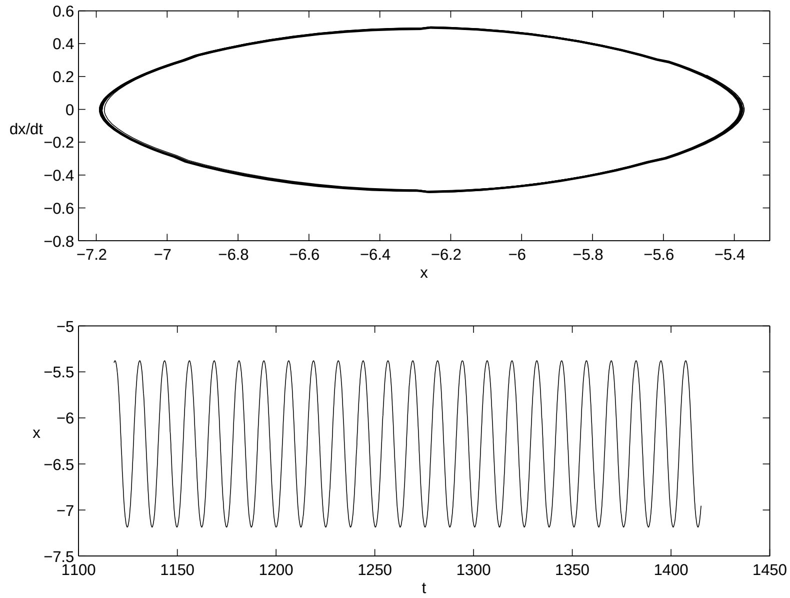

FIGURE 3. Phase portrait and solution $x(t)$ of the nonlinear pendulum $\ddot{x} + \sin x = \varepsilon\,[\gamma \cos \omega t - \alpha \dot{x}]$ with stabilizing perturbation $f^{*}(\omega, t) \approx -\delta(t)$. The parameter values are $\alpha = 0.1$, $\gamma = 6$, $\omega = 0.5$ and $\varepsilon = 0.1$.

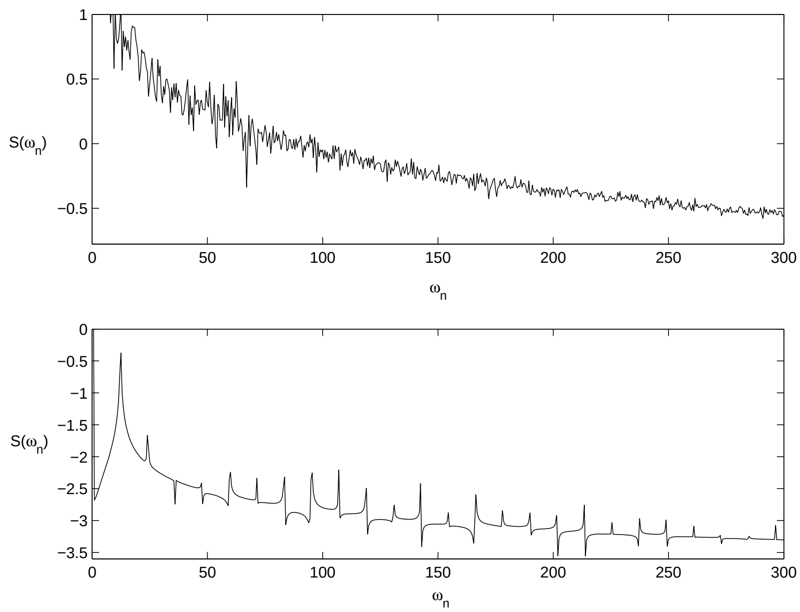

FIGURE 4. Power spectrum corresponding to the chaotic orbit of $\ddot{x} + \sin x + 0.01\dot{x} = 0.6 \cos 0.5t$ and to the orbit obtained after the stabilizing function acts for the pendulum.

5. **Conclusions.** Separatrix splitting is a very convenient method for examining dynamical systems, because it can be used to obtain non-integrability conditions for many applied problems in analytical form. Currently, the problem of chaos suppression considered in this study is mainly solved by numerical methods. However, asymptotic behavior of trajectories can be examined analytically and the distance between the invariant manifolds can be found in a general form by applying a perturbation method in the vicinity of a homoclinic trajectory.

In this study, separatrix splitting is applied to explore the possibility of chaos suppression in dissipative systems. As a result, we have provided a new mechanism of chaos suppression which allows us to find an analytical expressions for the external perturbation (stabilization function) that finally leads to the suppression of chaos. These results are sufficiently general to be applied to various dynamical systems that admit separatrix splitting.

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