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HIERARCHICAL MODELING OF A FORCED ROBERTS DYNAMO

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We investigate the dynamo effect in a flow configuration introduced by G. O. Roberts in 1972. Based on a clear energetic hierarchy of Fourier components on the steady-state dynamo branch, an approximate model of interacting modes is constructed covering all essential features of the complete system but allowing simulations with a minimum amount of computation time. We use this model to study the excitation mechanism of the dynamo, the transition from stationary to time-dependent dynamo solutions and the characteristic properties of the latter ones.

Keywords: Dynamo theory; bifurcation analysis; low-dimensional modeling.

1. Introduction

The generation of magnetic fields due to the motion of electrically conducting fluids is a well-studied phenomenon. Introducing an arbitrary weak seed field, one can observe either a decay, a conservation or even an amplification of this initial field. In the latter two cases, this behavior is usually called a dynamo effect [Moffatt, 1978], which is believed to be the physical reason for the occurrence of magnetism in planetary and astrophysical objects and has been realised in laboratory experiments [Gailitis *et al.*, 2000; Stieglitz & Müller, 2001].

The simplest mathematical framework for dynamo theory is given in terms of a system of nonlinear partial differential equations containing the Navier–Stokes and induction equations as coupled evolutionary equations for the fluid velocity $\mathbf{u}(\mathbf{x}, t)$ and the magnetic field $\mathbf{B}(\mathbf{x}, t)$. In a usual dimensionless form these equations read

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathrm{Re}^{-1}\nabla^2\mathbf{u} - \nabla p + (\mathbf{B} \cdot \nabla)\mathbf{B} + \mathbf{f}, \quad (1)$$

$$\partial_t \mathbf{B} + (\mathbf{u} \cdot \nabla)\mathbf{B} = \mathrm{Rm}^{-1}\nabla^2\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{u}, \quad (2)$$

where Re and Rm are the the kinetic and magnetic Reynolds numbers, p includes both, the hydrodynamic and the magnetic pressure, and $\mathbf{f}$ is an external body force. For simplicity, we restrict ourselves to the case of equal kinetic and magnetic Reynolds numbers (Re = Rm), i.e. the magnetic Prandtl number is fixed to the value Pm = 1 (for the configuration investigated in this study, effects of varying Pm were considered by Feudel *et al.* [2003]. Equations (1) and (2) are supplemented by the constraints $\nabla \cdot \mathbf{u} = 0$ and $\nabla \cdot \mathbf{B} = 0$.

In this paper, a model of a dynamo-active flow first studied by Roberts [1972] is further analyzed. The corresponding Roberts flow,

$$\mathbf{u}_R(x, y) = (\sin x \cos y,\ -\cos x \sin y, 2 \sin x \sin y)\,, \tag{3}$$

is obtained as a nonmagnetic solution of the Navier–Stokes equation (1) with the body force $\mathbf{f} = -\nabla^2 \mathbf{u}_R = 2\mathbf{u}_R$, which is also applied for simulations of nonlinear dynamos in the following. Continuing preceding investigations [Rüdiger *et al.*, 1998; Feudel *et al.*, 2003], we study the excitation mechanism of the dynamo and the character of different time-dependent dynamo solutions that bifurcate from a primary stationary one. As a detailed numerical bifurcation analysis with a high spatial resolution of the system is difficult due to high demands on numerics and computer capacity, we use an approximate model derived on the basis of the energy content of the respective Fourier modes and the symmetries of the problem which describes the essential properties and dynamical interactions of the full system correctly. In Sec. 2, this model is described in some detail. In Sec. 3, the excitation of the dynamo is studied with a special consideration of the energy transfer between the contributing Fourier modes. The essential features of the stationary and time-periodic dynamo regimes are also briefly discussed.

2. Description of the Model

The Roberts flow (3) is independent of the vertical (z) coordinate and periodic in the two horizontal directions x and y. We impose periodic boundary conditions with the same period of 2π in all three spatial directions. This specific setting motivates the use of Fourier series expansions $a(\mathbf{x}, t) = \sum_{\mathbf{k}} a_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}}$ of all dynamical variables such that all calculations can be completely performed in Fourier space. In accordance with the solenoidal property of $\mathbf{u}$ and $\mathbf{B}$, all Fourier coefficients are decomposed into components parallel and perpendicular to the respective wave vectors $\mathbf{k}$, resulting in a system of ordinary differential equations for the perpendicular components with the pressure term eliminated [Feudel *et al.*, 1996].

Several details of the bifurcation scenario of the full system[1] were already explored by Rüdiger *et al.* [1998] and Feudel *et al.* [2003]. Considering the kinetic and magnetic energy distributions $E_{\text{kin}}(\mathbf{k}) = 1/2\mathbf{u}_{\mathbf{k}} \cdot \mathbf{u}_{\mathbf{k}}^*$ and $E_{\text{mag}}(\mathbf{k}) = 1/2\mathbf{B}_{\mathbf{k}} \cdot \mathbf{B}_{\mathbf{k}}^*$ (asterisks denote the complex conjugates) over the different Fourier modes for the steady-state dynamo solution, we observe that the dynamics of the system is controlled by only a few dominating modes, i.e. there is a distinctive energetic hierarchy where only a small number of kinetic and magnetic modes is significantly excited (see Table 1).

For deriving an approximate model, the symmetries of the problem are particularly important. With periodic boundary conditions, the invariance of the Roberts flow and, thus, of the complete dynamical system with respect to translations in the z direction is described by the circle group S^1. Furthermore, the system possesses a discrete symmetry group including a subgroup $G = Z_4 \times_S D_2$ representing the geometric structure of the Roberts flow and the Z_2 invariance $\mathbf{u} \to \mathbf{u}$, $\mathbf{B} \to -\mathbf{B}$ as an intrinsic property of the MHD equations [Feudel *et al.*, 2003]. The discrete symmetry subgroup G survives the bifurcation, while the continuous S^1 invariance is broken (more precisely, the $S^1 \times Z_2$ symmetry is broken down to a pure Z_2 symmetry with respect to the transformation $\mathbf{u} \to \mathbf{u}, \mathbf{B} \to -\mathbf{B}$ combined with a shift by π in the z direction). In order to reproduce the correct bifurcation bahavior in an approximate model, with a given Fourier mode also all those modes have to be included which can be obtained from the given one by one of the discrete symmetry transformations. In this way, a hierarchy of models with increasing numbers of modes can be constructed.

A low-dimensional approximate model showing a dynamo bifurcation that breaks the S^1 symmetry and preserves the discrete subsymmetry consists of the following modes: The velocity field is still completely described by the original Roberts flow modes with $\mathbf{k} = (\pm 1, \pm 1, 0)$, which are only weakly disturbed in strength by the dynamo excitation and contain more than 99.9% of the total kinetic energy of the full system. Concerning the

[1] When referring to the full system, we consider the dynamic equations (1) and (2) (and their additional constraints) discretised with a high spatial resolution, corresponding to a consideration of all Fourier modes up to a certain cutoff wavenumber in each spatial direction.

Table 1. Kinetic and magnetic energy contributions of the different mode types on the steady-state dynamo branch for the approximate model (am) and the full model (fm) at Re = 3.0.

$\mathbf{k}$	$E_{\rm kin}^{(am)}$	$E_{\rm mag}^{(am)}$	$E_{\rm kin}^{(fm)}$	$E_{\rm mag}^{(fm)}$
$(\pm1, \pm1, 0)$	5.6605	—	6.213	—
$(0, 0, \pm1)$	—	0.0482	—	0.025
$(\pm1, \pm1, \pm1)$	—	0.1418	—	0.049
$(0, \pm2, \pm1)$	—	0.0204	—	0.010
$(\pm2, 0, \pm1)$	—	0.0204	—	0.010
$(\pm1, \pm3, \pm1)$	—	0.0176	—	0.005
$(\pm3, \pm1, \pm1)$	—	0.0176	—	0.005

magnetic field, the consideration of at least six different mode types with wave vectors $\mathbf{k} = (0, 0, \pm1)$, $(\pm1, \pm1, \pm1)$, $(\pm2, 0, \pm1)$, $(0, \pm2, \pm1)$, $(\pm3, \pm1, \pm1)$, and $(\pm1, \pm3, \pm1)$ is necessary to capture the essential features of the dynamical system, which is supported by the fact that the decrease in magnetic energy among the Fourier modes is not as strong as in the case of the velocity field. The approximate model studied in the following thus includes, taking into account the reality condition, $a^*_{\mathbf{k}} = a_{-\mathbf{k}}$, 19 independent complex Fourier modes. Since the solenoidal projections of $\mathbf{u_k}$ and $\mathbf{B_k}$ each correspond to two independent complex scalars per mode, this finally gives a system of $(19 \cdot 2 \cdot 4 =)$ 152 real differential equations.

3. Dynamo Bifurcations

The onset of dynamo action at a critical value of the Reynolds number is caused by the instability of a purely magnetic eigenmode with $k_z = \pm1$, i.e. with the largest possible wavelength in the z direction. Among the contributions to this eigenmode, the mean field mode with $\mathbf{k} = (0, 0, \pm1)$ (mean here refers to averages over the horizontal x and y directions) is most important for the physical mechanism of the Roberts dynamo. In accordance with earlier simulations of the full system, we find that at Re ≈ 2.61 (see Fig. 1), a degenerate pitchfork bifurcation with a twofold critical real eigenvalue takes place where the duplicity of the eigenvalue is directly related to the breaking of the continuous S^1 symmetry (the shift towards smaller values with respect to the full model, with a critical value of Re ≈ 2.7,[2] is explained by the decrease in dissipation with truncation). The two-dimensional unstable linear eigenspace consists of a continuum of modes where one can be transformed into the other by a translation in the z direction. Correspondingly, the bifurcating dynamo branch consists of a continuum of equivalent solutions describing a group orbit of the system which is represented by a zero real eigenvalue conserved for increasing Re.

The successive excitation of Fourier modes at the onset of dynamo action is well described by the approximate model. First, the unstable magnetic modes with $\mathbf{k} = (0, 0, \pm1)$, representing the mean field, interact with the Roberts flow to excite magnetic modes with $\mathbf{k} = (\pm1, \pm1, \pm1)$. The latter ones combine with the original Roberts flow to support in turn the mean field modes with $\mathbf{k} = (0, 0, \pm1)$ and, in addition, to excite magnetic modes with $\mathbf{k} = (0, \pm2, \pm1)$ and $(\pm2, 0, \pm1)$, which again interact with the basic flow resulting in excitations of the remaining model components (this mechanism is schematically displayed in Fig. 1). This successive excitation allows to understand the distinctive energetic hierarchy of Fourier modes in the full system as it continues in a cascade-like way. In a similar way, in the full system perturbations of the flow pattern occur due to Lorentz forces resulting from interactions of excited magnetic modes. However, as the Roberts flow modes are not directly involved here, these perturbations are extremely weak.

Subsequent to the pitchfork bifurcation, a Hopf bifurcation takes place at Re ≈ 3.24, after which all modes of the approximate model are excited both kinetically and magnetically. The imaginary part of the pair of critical eigenvalues, $\omega \approx 2.2$, corresponds to the frequency of the mode coefficents which are newly excited by this bifurcation and oscillate with a zero mean value. By contrast, the mode coefficients that were nonzero already on the steady-state branch start to oscillate with twice

[2]The numerical value differs from that given by Rüdiger *et al.* [1998] due to a different definition of Re.

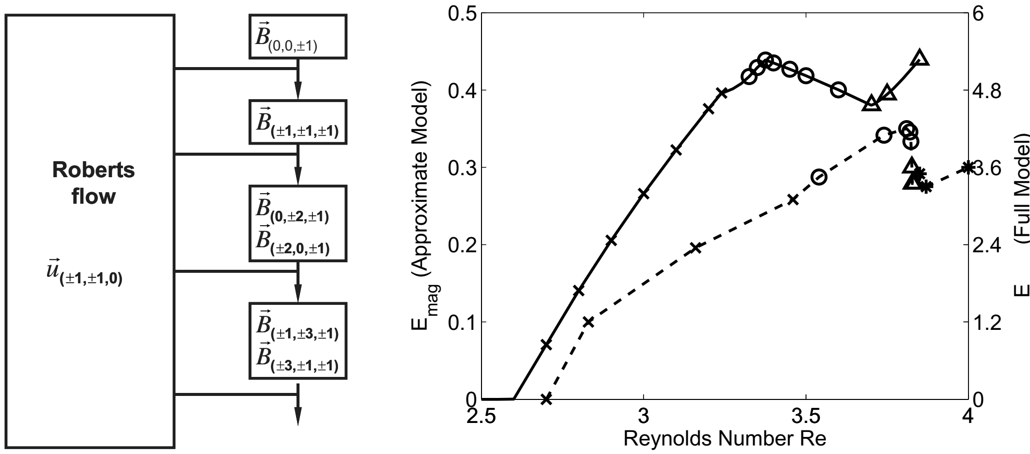

Fig. 1. (Left) Schematic drawing of the excitation cascade (vertical hierarchy from top to bottom) of Fourier modes in the approximate model. (Right) Magnetic energy of the approximate model (solid) and the full system (dashed), representing the bifurcation scenario of both systems. Symbols indicate the different dynamo states: stationary (×), time-periodic (∘), quasiperiodic (△) and chaotic (∗). The values for the full system are from [Rüdiger *et al.*, 1998], where a different normalization was used. Therefore, absolute values of the magnetic energy are not comparable for full and approximate systems.

the Hopf frequency. The resulting time-periodic dynamo solution shows a nontrivial spatio-temporal structure characterized by the surviving symmetry to the transformation given by the reflection $\mathbf{u} \to \mathbf{u}$, $\mathbf{B} \to -\mathbf{B}$ combined with a translation by π in the z direction and a phase shift by half a period ($T/2$). As a consequence, the modes are either symmetric under a time shift by $T/2$ or they are antisymmetric [Amdjadi & Gomatam, 1997]. The modes already excited in the stationary state before the Hopf bifurcation have to be symmetric to the spatial part of the transformation without the time shift, while the newly excited ones are lacking this purely spatial symmetry. Thus, two oscillation frequencies appear in the system and the modes which belong to the symmetric subspace oscillate with twice the Hopf frequency. The temporal behavior of the system corrsponds to a special kind of standing oscillation. Unlike a standard standing wave, where in a plane spanned by any two of the real or imaginary parts of the Fourier coefficients the system moves on a straight line, the trajectories in the planes spanned by the real and imaginary parts of a given Fourier coefficient are ellipses (with the exception of the modes making up the mean magnetic field, see below). As the $S^1 \times Z_2$ symmetry breaking is of the same nature as described by Amdjadi and Gomatam [1997] for $O(2)$ symmetric systems, we identify the state as oscillating waves.

Considering the energetic behavior as a function of Re, we observe that the average kinetic energy (not shown here) weakly decreases in the Reynolds number interval between the Hopf bifurcation and Re ≈ 3.37 before increasing again with higher values of Re, while the magnetic energy continues to increase after the Hopf bifurcation and then starts to decrease at Re ≈ 3.37 (see Fig. 1). The qualitative change of the system at Re ≈ 3.37, obviously a tertiary bifurcation, is mainly reflected by the modes making up the mean magnetic field, with $\mathbf{k} = (0, 0, \pm 1)$. The mean field $\langle \mathbf{B} \rangle$ generated in the (primary) dynamo bifurcation is force-free or a Beltrami field, satisfying $\nabla \times \langle \mathbf{B} \rangle = -\langle \mathbf{B} \rangle$. Thus, the modulus of the mean magnetic field is independent of z. In the time-periodic regime between the Hopf bifurcation and the tertiary bifurcation at Re ≈ 3.37, $\langle \mathbf{B} \rangle$ has still the Beltrami property, oscillating with spatially constant modulus. At Re ≈ 3.37 the temporal behavior of the mean-field modes becomes similar to that of the others namely, the projection of the system trajectory on the plane spanned by the real and imaginary parts of $\mathbf{B}_{(0,0,1)}$ changes from a straight line to an ellipse. The modulus of the mean magnetic field now oscillates in both, time and space and $\langle \mathbf{B} \rangle$ is no longer force-free, i.e. it exerts a direct Lorentz force on the fluid, which might be the reason for the decrease of the magnetic energy starting at Re ≈ 3.37. This state is stable up to about Re ≈ 3.7 where a transition to a quasiperiodic solution is observed which is replaced by a chaotic state at Re $\gtrsim 3.90$.

4. Conclusion

We have demonstrated that the dynamo action in a forced Roberts-type flow configuration is essentially provided by the interaction of the original flow with very few basic magnetic modes. In particular, a low-dimensional approximate model was derived containing the magnetic modes with wave vectors $\mathbf{k} = (0, 0, \pm 1)$, representing a horizontally averaged mean field, the modes with $\mathbf{k} = (\pm 1, \pm 1, \pm 1)$, which are coupled to the Roberts flow and the mean magnetic field through triadic mode interactions, and, in addition, four secondary magnetic mode types.

The approximate model reproduces all features of the stationary and the first time-periodic dynamo branches observed in the full system. The bifurcations to these states can be understood as essentially governed by the symmetries of the system. The spiral-staircase structure of the mean magnetic field and the character of the primary dynamo bifurcation coincide with the experimental results in the Karlsruhe dynamo facility designed with a similar flow pattern [Stieglitz & Müller, 2001].

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