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Chaos-induced resonant effects and its control

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Abstract

This Letter shows that a suitable chaotic signal can induce resonant effects analogous to those observed in presence of noise in a bistable system under periodic forcing. By constructing groups of chaotic and random perturbations with similar one-time statistics we show that in some cases chaos and noise induce indistinguishable resonant effects. This reinforces the conjecture by which in some situations where noise is supposed to play a key role maybe chaos is the key ingredient. Here we also show that the presence of a chaotic signal as the perturbation leading to a resonance opens new control perspectives based on our ability to stabilize chaos in different periodic orbits. A discussion of the possible implications of these facts is also presented at the end of the Letter.

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1. Introduction

In the last twenty years a big effort has been made to explore the constructive role of small perturbations added to the dynamics of bistable or multistable systems externally driven by either periodic or aperiodic forces. When modeling the response to external forcing of some complex systems characterized by two well-defined, stationary (or long-lived) states, it is worth considering the dynamics of the relevant variables by means of that of a classical particle moving in a symmetric double-well potential. We consider then that it is subjected to a weak sinusoidal signal, not strong enough to induce jumps between the two wells, that represents the main external forcing acting on the system. The rest of degrees of freedom of the whole system, or a new term being added to the main forcing by some experimenter, are supposed to be less relevant. Thus, they are usually added to the full dynamical description of the system by means of one or more small perturbative terms. In the theory of Stochastic Resonance (SR), for instance, this perturbation is supposed to be adequately described by means of a "noisy" term and then some random process, usually a Gaussian white noise, is used [1,2]. Rather surprisingly, in this situation a large variety of input–output coherence quantifiers show maximal values for finite values of noise intensity. Similar resonant effects have been observed in other systems in presence of noise, such as certain planar systems in absence of forcing [3].

But other situations can be envisaged, where there is a maximum in the input–output coherence without any external random contribution. That is the case of some chaotic systems [4–6] that might be close to a crisis [7,8], where the coordination between a periodic input and the output can be optimized by varying one of the system's parameters. Here we are more interested in the situation where the small perturbation arises from the chaotic dynamics of a subsystem weakly coupled to the relevant variables of the system. In this case we can speak of *Chaotic Resonance* (CR) [9–11]. Considering this, some authors have pointed out that in situations in which noise is supposed to play a role in generating a resonant phenomenon maybe a chaotic perturbation is at work.

In this work we give a step forward in this direction by analyzing the effect of chaos and noise in a simple nonlinear system. Among the references cited in the former paragraph, only

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in Refs. [9,10] the effect of both chaos and noise in a bistable system were evaluated, considering Gaussian white noise and logistic chaos. However, here we evaluate the effect of random and chaotic perturbations whose properties are similar, in a sense that will be precise later. From our explorations it becomes evident that chaos and noise can give rise to very similar levels of coherence between the periodic input and the output in a simple example, as long as we chose them to be "sufficiently similar".

In this context, the question of whether it is relevant or not to know if a resonance in a dynamical system is due to a chaotic perturbation or to noise arises naturally. From a fundamental point of view, this issue can be framed in the longstanding problem of the equivalence between chaotic and stochastic signals as activators of different process, recently reviewed in [12]. Here we show that this might also be relevant from a control point of view. In fact, we show that if the considered resonance is of chaotic origin new control perspectives arise. One of the most interesting properties of chaos is that it can be easily stabilized in a wide variety of periodic behaviors [13]. By making use of a simple example, we illustrate here that this property can also be of relevance in systems displaying CR, as long as it allows to design a control scheme that can either tame or enhance the input–output coherence without changing the perturbation intensity. A discussion of the contexts in which this idea might be useful is also given at the end of this Letter.

2. Our model

Throughout this Letter we use the following map as an example of a bistable dynamical system [15]:

$$x_{n+1} = S\tanh(x_n) + \epsilon\sin(2\pi n/T_0) + \alpha\xi_n, \tag{1}$$

where S, T_0 and α are parameters. We fix $S = 2$, so if $\epsilon = 0$ and $\alpha = 0$ the system possesses two stable fixed points, as we can see in Fig. 1. To such an autonomous system we add a periodic forcing with $\epsilon = 0.5$, unable to induce jumps between positive and negative values of x, and whose period is $T_0 = 32$. In order to complete the analogy with the typical system displaying SR, we add an external perturbation ξ_n. Here we mainly focus on the role played by ξ_n, whose intensity is modulated through the parameter α. This perturbation can either be random or chaotic,

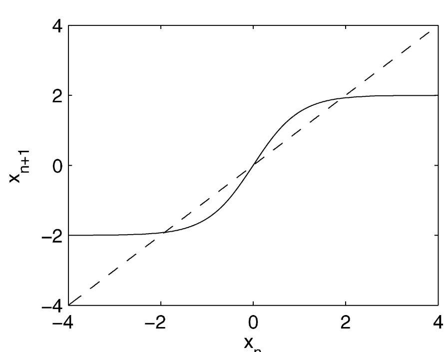

Fig. 1. Graph of a simple bistable dynamical system, the map $x_{n+1} = 2\tanh(x_n)$, for which there are two stable fixed points.

and hence it may obey a deterministic law $\xi_{n+1} = f(\xi_n)$. Another type of *chaotic* perturbation that we consider here is given by the quantity $\xi_n = g(\xi_n^1, \xi_n^2)$ that is constructed from two chaotic systems verifying $\xi_{n+1}^j = f(\xi_n^j)$, $j = 1, 2$.

We present a novel way to analyze the resonant phenomena caused by either a chaotic or a random perturbation, in order to see more precisely which is the specific effect played by the different origin of these perturbations. In previous works [9,10] the effect of a chaotic perturbation and Gaussian white noise in a bistable dynamical system were analyzed. However, these perturbations have very different statistical properties. Thus, here we construct groups of chaotic and random perturbations in such a way that they have similar statistical properties at the level of one-time distribution functions, and then we compare their effect on our system. This comparison is done by evaluating a quantifier of the input–output coordination as a function of the perturbation intensity $D = 2\sigma^2$, where σ is the standard deviation of the perturbation [2]. The dependence of σ on α is given by the following relation:

$$\sigma^2 = \alpha^2 \int_I \big(\xi - \langle\xi\rangle\big)^2 p(\xi)\, d\xi, \tag{2}$$

where I is the interval to which ξ is restricted, $p(\xi)$ is either the probability density for the random perturbations or the density of the measure [16] for the chaotic perturbations and $\langle\xi\rangle$ is its mean value.

3. Generating similar chaos and noise

In this section we describe how we have obtained the groups of similar chaotic and random perturbations that we apply to the system described in the former section.

It is well known that the tent map $\xi_{n+1} = 1 - 2|\xi_n|$ [16] generates a chaotic signal whose density of the measure is $p(\xi) = 1$, for $\xi \in [-0.5, 0.5]$, see Fig. 2. So does the Bernoulli shift map, $\xi_{n+1} = 2\xi_n + 0.5$ if $\xi \in [-0.5, 0)$ and $\xi_{n+1} = 2\xi_n - 0.5$ for $\xi \in [0, 0.5]$. An easy way to obtain a random perturbation with the same one-time statistical properties is to

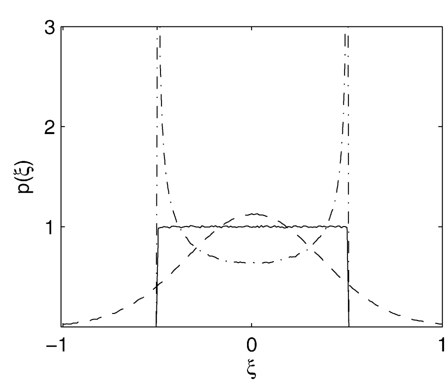

Fig. 2. Probability distributions and density of the measure for some perturbations, computed numerically. Density of the measure of the tent map, showing a uniform probability distribution in $[-0.5, 0.5]$ (—); Probability density of the logistic noise perturbation, following the same statistics as the trajectories of the logistic map (·-·-); Probability distribution of the chaotic Gaussian perturbation, with an analogous appearance as that of a Gaussian noise (- - -).

generate numbers in the $[-0.5, 0.5]$ interval with a uniform probability distribution. Then we have a first group of similar chaotic and random perturbations, that we call *tent perturbations*.

We do also generate a chaotic perturbation for our system by using the logistic map $\xi_{n+1} = 0.5 - 4\xi_n^2$, as well as the piecewise continuous modified logistic map: $\xi_{n+1} = 0.5 - 4\xi_n^2$ for $\xi \in [-0.5, 0)$ and $\xi_{n+1} = 4\xi_n^2 - 0.5$ for $\xi \in [0, 0.5]$. Although for other values of the parameter these two maps also present chaotic dynamics, for this one the explicit form of the density of the measure is well known $p(\xi) = 1/(\pi\sqrt{(0.5-\xi)(0.5+\xi)})$ [16]. The main advantage of knowing the explicit form of $p(\xi)$ is that there is a variety of algorithms that allow to obtain random signals with certain probability distribution [17]. However, here we have used the topological equivalence existing between the tent and the logistic map and also between the Bernoulli shift map and the modified logistic map in order to obtain the desired random perturbation imitating the statistics of the logistic map. The trajectories of the tent map are related one-to-one with those of the logistic map via the mapping $C(x) = \sin(\pi x)/2$ [16]. So, if we have a sufficiently large number of trajectories of the tent map which pass with the same probability through every point in the $[-0.5, 0.5]$ interval, they correspond one-to-one to trajectories of the logistic map passing through every point of the same interval and following the density of the measure of the logistic map. Then, for a given set of points η_i generated randomly and with uniform probability in the $[-0.5, 0.5]$ interval, the points $\xi_i \equiv C(\eta_i)$ have a probability distribution equal to the density of the measure of the logistic map, as shown in Fig. 2. Thus, we have a couple of chaotic perturbations, the logistic map and the modified logistic map, and a novel noisy perturbation that imitates it, the *logistic noise*. These three perturbations form the *logistic perturbations* group.

We can also generate a signal of chaotic origin that mimics the probability distribution of the Gaussian white noise. It is well known that if we generate two random numbers ξ_1 and ξ_2, uniformly distributed in $[0, 1]$, then the numbers $\xi = \sigma\sqrt{-2\log\xi_1}\cos(2\pi\xi_2)$ follow a Gaussian probability distribution with zero mean and standard deviation σ [17]. Thus, it is easy to see that if ξ_n^1 and ξ_n^2 are iterations of the tent map as defined above, whose values are uniformly distributed in the $[-0.5, 0.5]$ interval, the numbers $\xi_n = \sqrt{-2\log(\xi_n^1 + 0.5)}\cos(2\pi(\xi_n^2 + 0.5))$ also follow a Gaussian distribution, as we can see in Fig. 2. This is our new *chaotic Gaussian* perturbation. We refer to these two perturbations as the *Gaussian perturbations*.

Fig. 2 show clearly how different are statistically the three groups of perturbations. Thus, we consider more natural to evaluate the effect of each group of perturbations separately. We must stress, however, that the random signals that we generate are always white, so the time correlations of the chaotic and random perturbations of each group are very different. However, we will show that in some cases the similarities between the signals of each group are enough to make them induce nearly indistinguishable resonant effects. The results are shown in next section.

4. The effect of the perturbations

Once that we have described how to generate our groups of similar chaotic and random perturbations, we analyze how each of them acts in our system. As we mentioned earlier, these perturbations induce jumps between positive and negative values of x_n, and we want to evaluate the coordination between the forcing and the jumps as a function of D for each one of them. To do so, we use integration of the Residence Time Distribution Function (RTDF) [6,14,15] as described later, because it matches better with the intuitive picture of the resonances that we try to draw here. If we denote by t_i the times in which those jumps take place, the normalized distribution $N(T)$ of the quantities $T(i) = t_i - t_{i-1}$ is called RTDF. This distribution is known to show peaks centered at $T_k = (k - \frac{1}{2})T_0$. In analogy with Refs. [14,15] we define the areas under the different peaks as

$$P_k = \sum_{T_k - T_0/4}^{T_k + T_0/4} N(T). \tag{3}$$

Thus, the $k = 1$ value, P_1, as a function of the perturbation intensity D gives us a numerical quantifier of the coordination between the jumps and the forcing. The resonances can be identified by a maximum of P_1, and thus of the coordination, for certain values of D.

The results of the calculations performed for the groups of perturbations described above are depicted in Fig. 3. We start by making some comments on Fig. 3(a), where the results for the tent perturbations are shown. Notice that there is a clear non-monotonicity of the resulting curve for each of the three perturbations of this group, so there is a resonance independently of the random or chaotic nature of the perturbations. The maxima occur for approximately the same values of D, although the shapes of the curves obtained for each perturbation are quite different. However, the maximum of the curve obtained for the Bernoulli map case is higher than when using uniformly distributed random numbers. This example suggests that there are situations where the coordination between the jumps and the forcing in presence of chaos can be better than in presence of noise.

In the case of the logistic perturbations, Fig. 3(b), and in the case of the Gaussian perturbations, Fig. 3(c), there is also a resonance no matter whether the perturbation involved is chaotic or random. Furthermore Figs. 3(b) and 3(c) do also show that the curve obtained for logistic chaos is very similar to the one obtained for logistic noise, and similarly the curve generated by the chaotic Gaussian perturbation is very similar to the one obtained for Gaussian white noise. Thus, the responses of our system to the external perturbation in the chaotic and in the random case are nearly identical. In our opinion, this result reinforces the established idea of chaos as a plausible source of resonances in dynamical systems, as long as these examples show that chaotic perturbations can generate resonant responses very similar to those generated by noise.

As a final remark, we must say that it would have been possible to change the time scale of the chaotic perturbations by taking more than one iteration of the maps described in Sec-

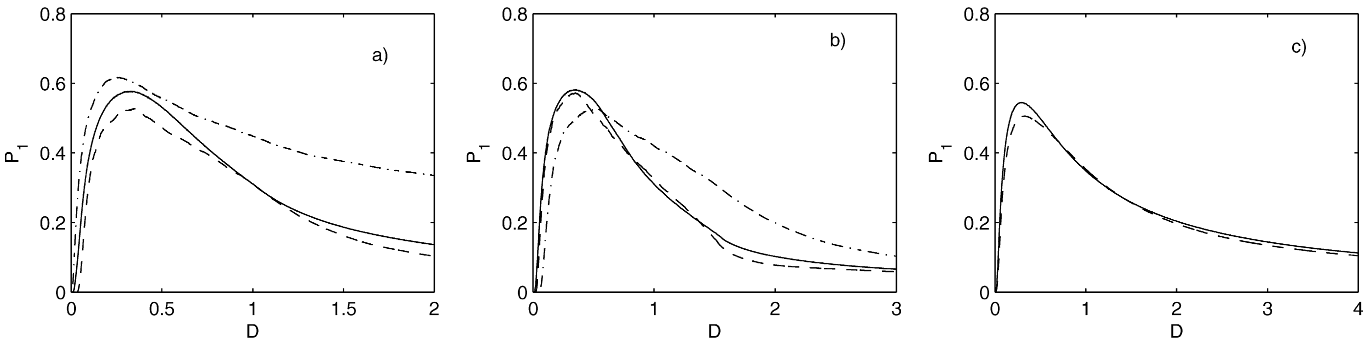

Fig. 3. The area of the peak P_1 as a function of the perturbation intensity D for different perturbations: (a) Tent chaos (- - -), chaos from the Bernoulli shift map (-·-·) and random numbers uniformly distributed in $[-0.5, 0.5]$ (—); (b) Logistic chaos (- - -), chaos from the modified logistic map (-·-·) and logistic noise (—). (c) Chaotic Gaussian perturbation (—) and Gaussian white noise (- - -). For all the perturbations there is a resonance, and in some cases the curves obtained for chaotic and random perturbations are very similar.

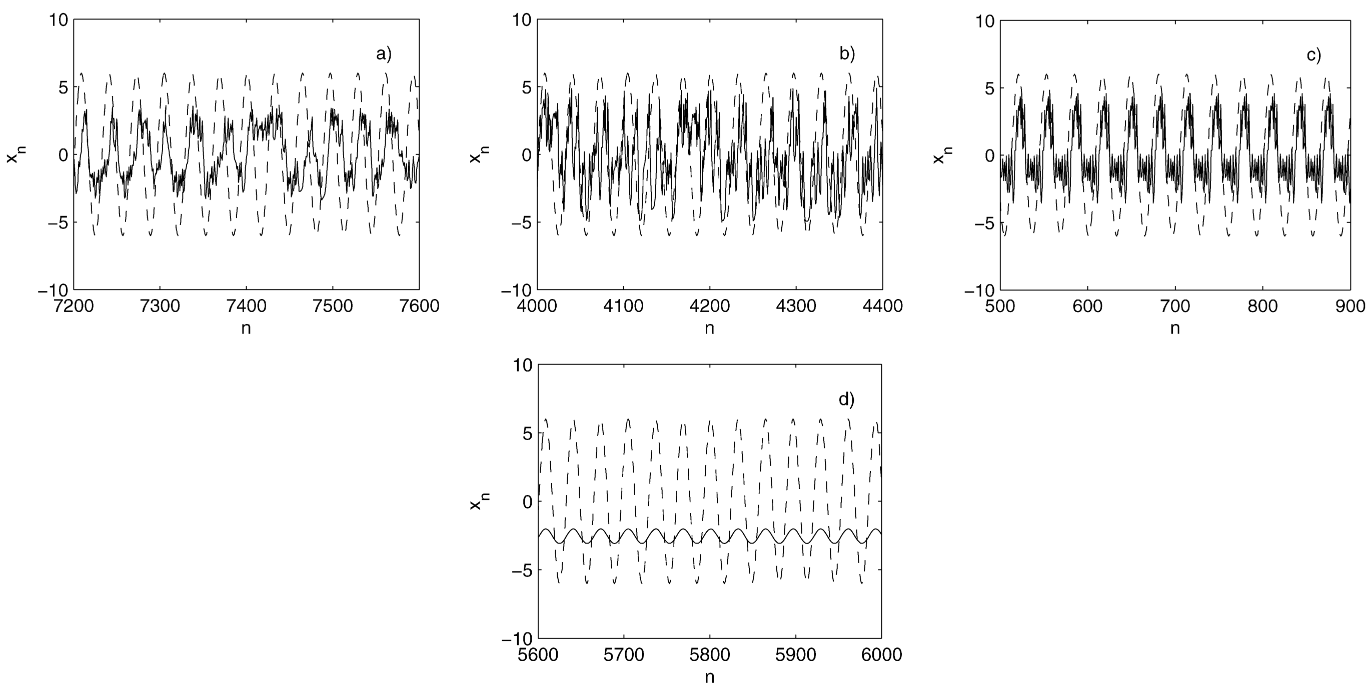

Fig. 4. Three time series of the bistable system perturbed with different signals generated with the logistic map, plotted with the periodic forcing (- - -), amplified in order to evaluate their coordination. (a) Time series of the system perturbed with iterations of logistic chaos for the optimal intensity, $D \approx 0.29$, showing a good coordination between the jumps of x_n and the forcing. (b) Time series for the system perturbed with logistic chaos again with $D = 1.5$, so the coordination is sensibly worse. (c) Time series for the system perturbed with logistic chaos stabilized in a period T_0 orbit with the same α that in the former case, so input–output coherence is greatly improved. (d) Time series for the system perturbed with logistic chaos with the same α as in the $D \approx 0.29$ case. However, here we have stabilized chaos in a period-one orbit, so instead of the jumps coordinated with the forcing that could be observed in absence of control here there are no jumps at all.

tion 3. This would have given us "faster" chaotic perturbations with the same one-time distribution function that the one obtained by taking just one iteration of the map (due to the invariance of the measure [16]). We expect that the similarities between the resonant effect induced by these faster chaotic perturbations and their random counterpart would be even clearer. Analogously, it would have also been possible to consider the effect of colored random signals, as long as it is well known that the resonances are sensitive the time correlation of the perturbation [18]. However, Fig. 3 shows that we do not need to tune neither the time scale of the chaotic perturbation nor the time correlations of the random perturbations in order to obtain nearly the same resonant effects. Thus, we have opted to restrict ourselves to the simpler case.

5. Control of chaotic resonance

In last section we have given evidences showing that chaos and noise can give rise to very similar resonant responses. In this section we are going to illustrate one of the main implications of having CR instead of SR in a dynamical system. We are going to show that the presence of chaos as the perturbation leading to a resonant behavior can have important consequences from a control point of view.

The good performance of a dynamical system displaying resonances usually lies on the degree of coordination between the periodic input and the evolution of one of the significant variables of the system, specially on how correlated are certain threshold crossings of that variable with the periodic forcing. In the simple case that we are considering here, such threshold is $x = 0$ (and each threshold crossing is called a jump). Thus, some schemes have been designed to control SR (some references can be found in [19]). Here we want to point out that, if the perturbation involved in the resonance is chaotic, new control possibilities arise. These possibilities are based in the well-known adaptability of chaotic systems, whose trajectories can be easily led to periodic motions by using just small perturbations [13]. Thus, in systems displaying CR we can take advantage of our ability to stabilize the source of the perturbation that induces the resonance in different periodic orbits.

This idea can be illustrated by considering our example when ξ_n is a chaotic signal from the logistic map. If we have a perturbation intensity $D \approx 0.29$ close to the optimal one (see Fig. 3(b)) the jumps of x_n between positive and negative values are quite coherent with the periodic input, as shown in the sample of Fig. 4(a). Instead, if the perturbation intensity takes a value far from the optimal one, for example $D = 1.5$, then jumps take place much more often and the coherence is worse, as shown in Fig. 4(b). However, in that situation we can improve the input–output coordination without varying α, the parameter that modulates the perturbation intensity, just by stabilizing the dynamical system $\xi_{n+1} = f(\xi_n)$ in one of the period-T_0 orbits that lie in the chaotic attractor, using the well-known OGY method [13]. This leads to a nearly perfect coordination between the jumps of x_n and the periodic forcing, as seen in Fig. 4(c).

It is also easy to see that, if we were interested on destroying the coordination between the jumps of the x_n and the input, we could just stabilize the system $\xi_{n+1} = f(\xi_n)$ in one of the period-1 orbits that lie in the attractor. For example, in Fig. 4(d) we are considering again the system perturbed with logistic chaos with the same α as in the $D \approx 0.29$ case, but now the logistic chaos has been stabilized in a period-one orbit so x_n takes always negative values. Thus, instead of observing the nice coordination between the jumps and the forcing that was observed in the uncontrolled case (Fig. 4(a)), we are able to tame those jumps. But all this would not be possible if ξ_n is purely random.

Finally, we must point out that the application of the control scheme will greatly depend on the type of dynamical system where the resonance is observed. More precisely, it will strongly depend on the coupling with the chaotic subsystem generating the chaotic perturbations and the structure of its chaotic attractor. However, we consider that the fact that chaos can be easily redirected to a large variety of dynamical behaviors with small perturbations can always be used to tame or enhance the resonances in these kind of systems. We must finally point out that the results shown in this section could have also been reproduced by making use of any of the chaotic perturbations considered in this paper instead of a chaotic signal from the logistic map.

6. Conclusions

In conclusion, in this work we have given further evidence showing that chaos plays a role analogous to noise in inducing a resonant behavior in a simple system. Most importantly, by using similar chaotic and random perturbations we have given strong evidences reinforcing the idea according to which, in some cases, it is very difficult to distinguish whether a resonant behavior is due to the effect of noise or to a chaotic perturbation. This is especially interesting, because it may imply that in real systems where noise is supposed to play a major role in the enhancement of periodic signals perhaps a chaotic signal is the main cause of this enhancement. Resonance-like behavior are thought to be connected with certain cooperative phenomena in neurons, like epilepsy, and some authors [6] have speculated that there might be a link between the evidence of chaotic activity in neural processes [20] and the occurrence of resonances in some simple dynamical systems in absence of random perturbations. Our work contributes then to make the link between chaotic dynamics and resonances more plausible. On the other hand, we have also illustrated that, if the perturbation is chaotic, a suitable chaos control scheme can enhance or tame drastically the input–output coherence. This can also be of great importance in the context mentioned above.

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