

## Phase control of excitable systems

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**Abstract.** Here we study how to control the dynamics of excitable systems by using the *phase control* technique. Excitable systems are relevant in neuronal dynamics and therefore this method might have important applications. We use the periodically driven FitzHugh–Nagumo (FHN) model, which displays both spiking and non-spiking behaviours in chaotic or periodic regimes. The phase control technique consists of applying a harmonic perturbation with a suitable phase  $\phi$  that we adjust in search of different behaviours of the FHN dynamics. We compare our numerical results with experimental measurements performed on an electronic circuit and find good agreement between them. This method might be useful for a better understanding of excitable systems and different phenomena in neuronal dynamics.

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## 1. Introduction

The paradigmatic FitzHugh–Nagumo (FHN) system [1] has been broadly investigated in the past. In spite of its mathematical simplicity, many effects observed in neuronal cells are qualitatively contained in it [2]. In particular, the periodically driven FHN system has been used to investigate the effects of external perturbations on the generation of electrical pulses in neurons. More precisely, the periodically driven FHN model has been used recently to investigate the role of noise in the encoding process, i.e. in the relation between the dynamical response of the FHN neuron model and the external driving [3, 4].

Although this is an important issue from a neuronal dynamics point of view, there is a lack of inquiries on how the input–output relation could be controlled by an external perturbation. A number of techniques have been proposed to control the dynamics of different models related with the FHN system [5]–[8]. Considering that the periodically driven FHN presents chaotic dynamics, the control of this system can be tackled from a chaos control point of view [9]. The methods to control chaos can be classified into *feedback* and *nonfeedback methods* [10, 11]. Feedback methods, like the paradigmatic methods described in [12, 13], stabilize one of the unstable orbits that lies in the chaotic attractor by applying small state-dependent perturbations to the system. However, for experimental implementations, these methods require the application of a fast and adequate response to the system that sometimes might not be available. In such cases, nonfeedback methods are more useful.

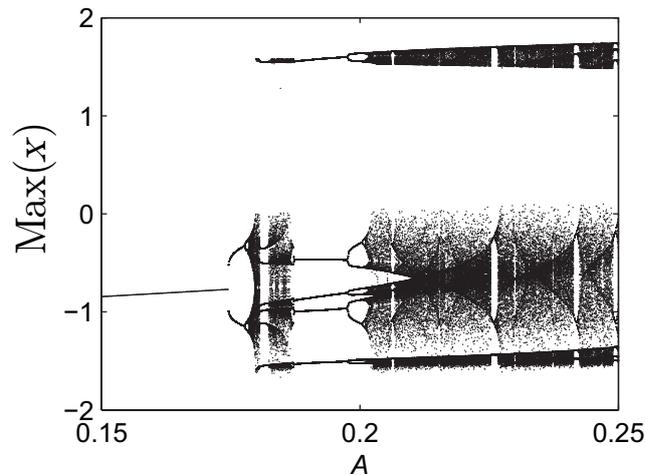
An important class of nonfeedback methods is that based on applying a harmonic perturbation either to some of the parameters of the system or as an additional driving; and its effectiveness was discussed numerically and experimentally in different works [14, 15]. These types of nonfeedback methods have been mainly used to suppress chaos in periodically driven dynamical systems. A paradigmatic family of systems of this type are nonlinear oscillators, whose equations can be written as

$$\ddot{x} + \mu\dot{x} + \frac{dV}{dx} = F \cos(\omega t), \quad (1)$$

where  $\mu$  is the damping coefficient,  $V(x)$  the potential function and  $F \cos(\omega t)$  an external periodic driving. Depending on the potential  $V(x)$  we have different kinds of oscillators. The dynamical system described in (1) can be written as a system of two coupled first-order differential equations with a periodic driving. Such is the prototype model considered in this paper, the periodically driven FHN model.

In [15], it was observed that when these types of nonfeedback methods were used to control the dynamics of a periodically driven chaotic system, the phase difference  $\phi$  between the periodic driving and the perturbation had a great influence in the dynamical behaviour of the system. Furthermore, Qu *et al* [16] show that  $\phi$  plays a crucial role in the global dynamics of a well-known nonlinear oscillator, the Duffing oscillator. This control technique is called *phase control of chaos* and it has been extensively explored in [17]. Besides chaos control, the phase control has been used to control crisis-induced intermittency [18] and to control escapes in open dynamical systems [19].

In this paper, we apply this method to a paradigmatic excitable system, the periodically driven FHN system, which is one of the most utilized to study the spiking activity of a neuron [2]. We show that this control method allows one to control different aspects of the dynamics of the system, that is, to tame or enhance the spiking activity as well as to control chaos. Thus, this work shows that the phase control method can be used to control a type of



**Figure 1.** Numerical bifurcation diagram for the maximum of the voltage  $u$  by taking as parameter the amplitude of the external driving  $A$ . A spiking regime is observed for high values of  $A$ , whereas for low values we are in the non-spiking regime.

dynamics that was not considered in previous applications of this technique [15], [17]–[19]. Since the periodically driven FHN system is a paradigmatic model in neuronal dynamics, the results obtained in this work can be, in principle, generalized to higher dimensional problems such as the Hodgkin–Huxley model [20].

This paper is organized as follows. In section 2, we present the FHN model and we describe the implementation of the phase control scheme for this system. Section 3 presents numerical simulations showing that the phase control technique can both tame or enhance the spiking regime of the FHN and suppress its chaotic behaviour. Finally, we give experimental evidence of the validity and robustness of this method by implementing it in an electronic circuit, as described in section 4. Discussion and conclusions of the main results of this paper are presented in section 5.

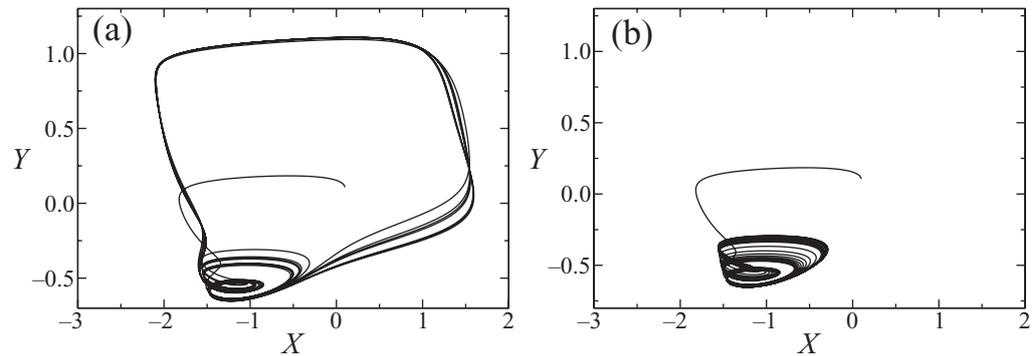
## 2. Model description

The FHN system is governed by the following equations:

$$\begin{aligned} \frac{dx}{dt} &= c(-y + x - (x^3/3) + S(t)), \\ \frac{dy}{dt} &= x - by + a, \end{aligned} \quad (2)$$

where, in the context of neurophysiology,  $x(t)$  is called the voltage variable,  $y(t)$  the recovery variable,  $S(t) = A \sin(\frac{2\pi}{T}t)$  is an external driving of period  $T$  and amplitude  $A$  and  $a, b, c$  are real positive constants.

By increasing the amplitude of the external driving, the system undergoes a period doubling bifurcation and for sufficiently high values of  $A$  a spiking regime is observed, as shown in figure 1, where the maximum values of  $x$  are represented as a function of  $A$  (the spiking



**Figure 2.** Numerical trajectories in the phase space with the same initial condition for the periodically driven FHN model in (a) the spiking regime and (b) the non-spiking regime.

starts with the onset of the upper branch, with  $x$  around 1.5). The set of parameters used in this bifurcation diagram as well as in the rest of the numerical simulations is  $a = 0.7$ ,  $b = 0.8$ ,  $c = 12.5$  and  $T = 1.125$ .

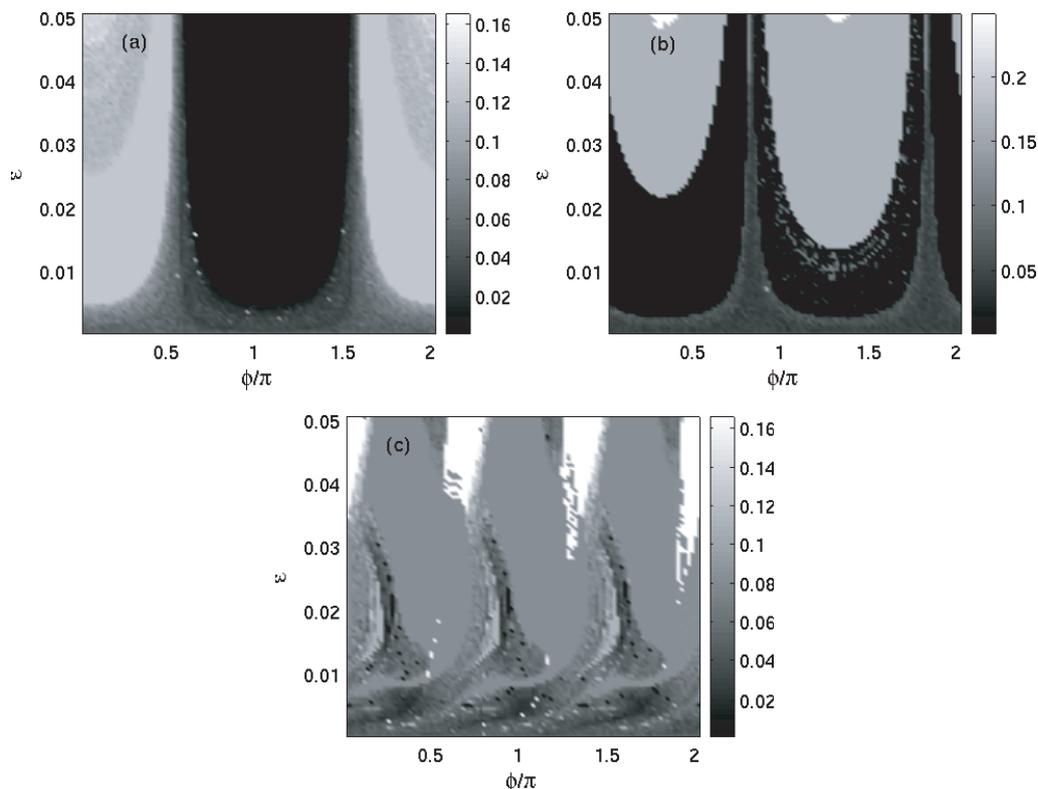
From figure 1, we can distinguish two different behaviours, spiking ( $A \geq 0.1798$ ) and non-spiking ( $A < 0.1798$ ). Figures 2(a) and (b) show, respectively, numerical trajectories in the phase space corresponding to these different regimes. Notice that in both cases the dynamics of the system is chaotic.

In our phase control scheme, we add some small harmonic or perturbation to one of the accessible parameters of the system. Here, we add a small periodic perturbation to the amplitude of the external driving  $S(t)$ . Then, the term  $S(t)$  present in (2) is replaced by the term  $S'(t) = A(1 + \delta(t)) \sin(\frac{2\pi}{T}t)$ , where  $\delta(t) = \epsilon \sin(\frac{2\pi r}{T}t + \phi)$ . The parameter  $r$  is the ratio of the frequency of the applied perturbation  $\delta(t)$  and the frequency of the external driving  $S(t)$ . The key parameter for our control scheme is  $\phi$ , the phase difference between the applied perturbation and the driving. The phase control scheme relies on an appropriate selection of the phase  $\phi$ , once  $\epsilon$  and  $r$  are fixed, in order to lead the system to the desired dynamical regime. In the next section, this idea will be tested numerically.

### 3. Numerical results

Now we are going to consider the influence of the phase control method on the dynamics of the system considered. We must point out that we are interested in the effect of this control scheme both when the system is in the chaotic spiking regime, as in figure 2(a), and when it is in the chaotic non-spiking regime, as in figure 2(b). With this purpose, we will evaluate the effect of phase control when the driving amplitude of the unperturbed system is  $A = 0.1802$ , which leads to a chaotic spiking regime, and for  $A = 0.17$ , for which the system is in a chaotic non-spiking regime, as we can see from figure 1.

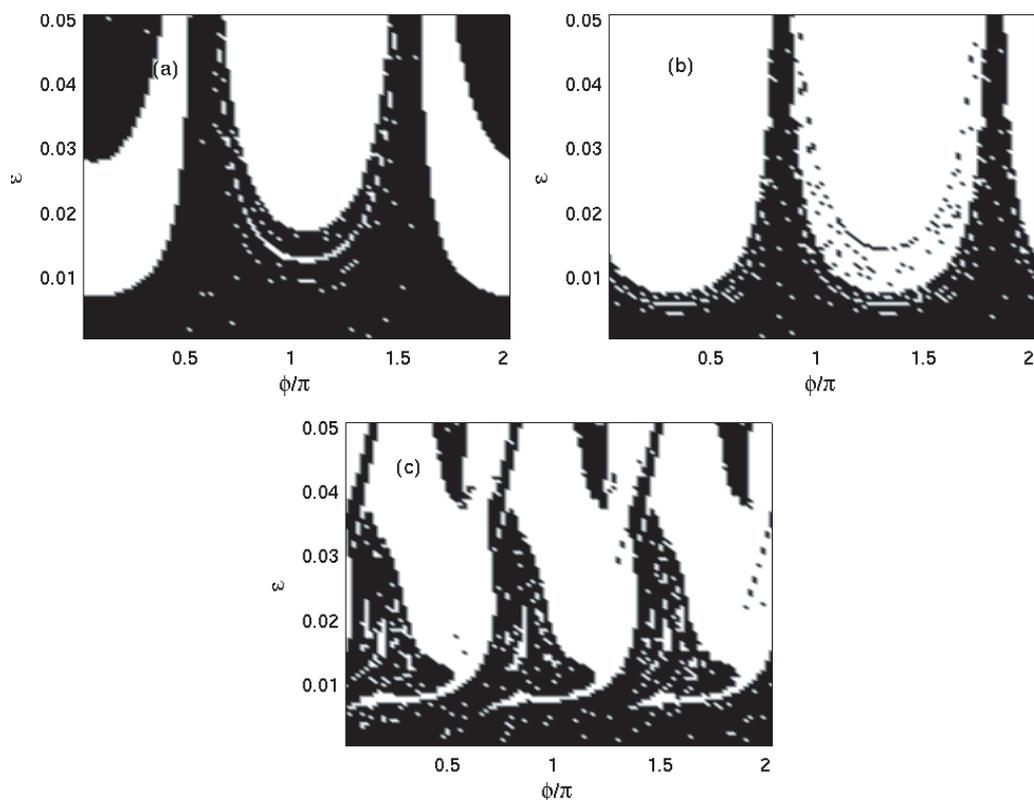
Since we are dealing with an excitable chaotic system, we must consider different effects of the phase  $\phi$  on the dynamics, for fixed values of  $\epsilon$  and  $r$ . The first relevant issue in this context is whether the control can either tame or enhance the spiking regime. This can be done numerically as follows. Considering the picture shown in figure 2(a), we can notice that a clear feature of the spiking regime is that the  $x$  variable of the system crosses the threshold value



**Figure 3.** Number of spikes per cycle  $H$  obtained by applying the phase control to the FHN in the spiking regime, as a function of  $\epsilon$  and  $\phi$  for (a)  $r = 1$ , (b)  $r = 1/2$  and (c)  $r = 1/3$ . The colour code refers to the  $H$  value.

$x = 1$ , whereas in the non-spiking regime such crossings do not occur. Thus, a way to know if the system is in the spiking regime or not, is to evaluate the number  $H$  of spikes per cycle (of the main driving). This quantity can be computed by counting the number of threshold crossings of the  $x$  variable (with  $\dot{x} > 0$ ) in a long time series and dividing this result by the number of cycles of the main driving  $S(t)$ . Note that with this definition the average spiking frequency of the system is the product of  $H$  times the frequency of the main driving. For sufficiently long time series, we expect  $H$  to converge to a fixed value. In the non-spiking regime, we expect to obtain  $H = 0$ . On the other hand, following the tradition of control of chaotic systems, we also want to evaluate if our control method can regularize the system dynamics, i.e. if an adequate selection of the phase  $\phi$  for different values of  $\epsilon$  and  $r$  can turn the chaotic motion into periodic. In order to do this, we calculate the largest Lyapunov exponent (LLE) of the system.

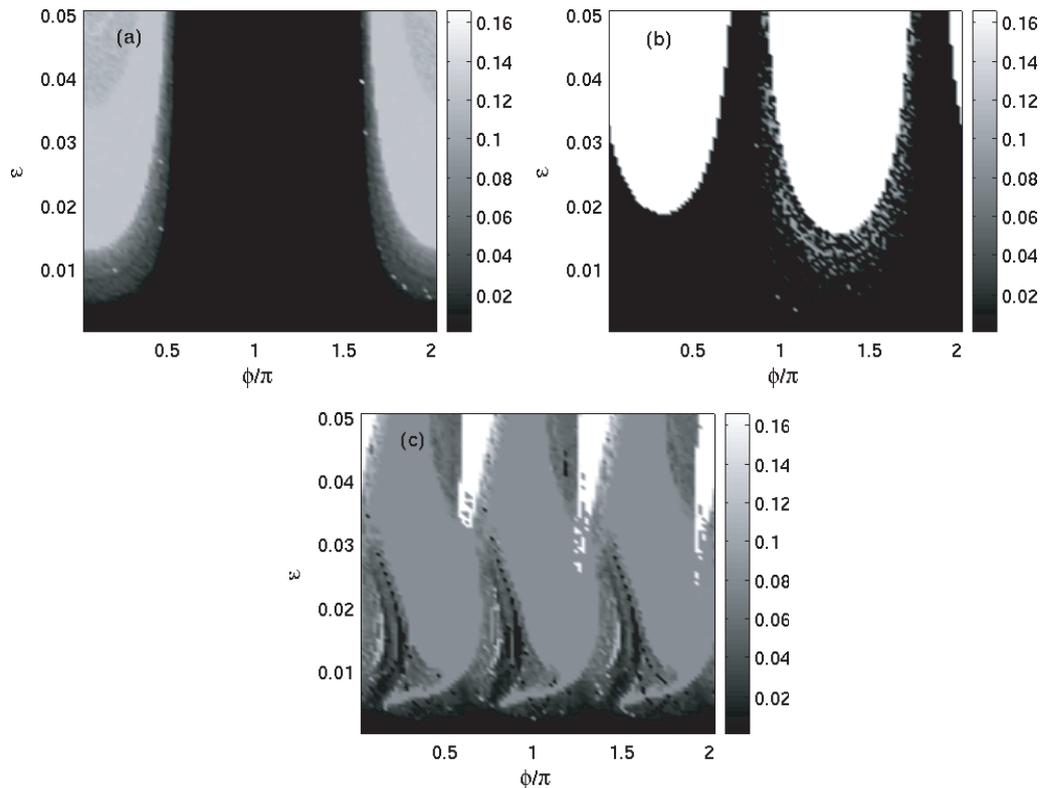
We first consider the effects of the phase control when FHN is in the spiking regime. The values of  $H$  when phase control is applied are shown as a function of  $\epsilon$  and  $\phi$  in figure 3 for  $r = 1$ ,  $r = 1/2$  and  $r = 1/3$ . We notice that for  $r = 1$  and  $r = 1/2$  the phase control can tame the spiking regime, i.e. an adequate selection of  $\phi$  can tame the spiking for small values of  $\epsilon$ . For  $r = 1/3$ , though, taming the spiking regime is not possible. An interesting feature that arises from these calculations is that an adequate selection of  $\phi$  can also increase the number of spikes  $H$  per cycle obtained. It is clear that for certain combinations of  $\phi$  and  $\epsilon$  the computed value of  $H$  is sensibly larger than the value that would be obtained in the absence of perturbation ( $\epsilon = 0$ ). This feature is present in all the numerical simulations performed.



**Figure 4.** Lyapunov exponent obtained by applying the phase control to the FHN in the spiking regime, as a function of  $\epsilon$  and  $\phi$  for (a)  $r = 1$ , (b)  $r = 1/2$  and (c)  $r = 1/3$ . Black colour denotes positive values of LLE (chaotic regime) and white colour negative values of LLE (periodic regime).

Figures 4(a)–(c) give the LLE that results when we apply our control scheme to this system. As in the previous calculations, we represent the LLE values as a function of  $\epsilon$  and  $\phi$  for  $r = 1$ ,  $r = 1/2$  and  $r = 1/3$ , respectively. The similarities between this figure and figure 3 are not surprising, considering that a change in the type of dynamics typically implies a change in the value of the LLE. We can observe that for all  $r$ , and for small  $\epsilon$ , an adequate selection of  $\phi$  can lead the system to a periodic state. On the other hand, by combining these results with the results observed in figure 3, we can see that the enhancement of the number of spikes per cycle,  $H$ , obtained for different combinations of  $\epsilon$  and  $\phi$  is due to a regularization of the system dynamics, as long as they correspond to combinations of  $\epsilon$  and  $\phi$  for which the dynamics is spiking and periodic. Finally, we must point out that bifurcation diagrams performed show that the transitions between chaotic and periodic motion typically occur by inverse period-doubling bifurcation, which is the typical mechanism arising in this control scheme [16, 17].

We focus now on the effects of the phase control scheme on the non-spiking regime. We start by evaluating the number of spikes per cycle  $H$  observed when applying the control to the system for different combinations of  $\epsilon$  and  $\phi$ , shown in figure 5. We can observe here again that for  $r = 1$  and  $r = 1/2$ , and for an adequate value of  $\epsilon$ , the selection of  $\phi$  will determine whether the system is in the non-spiking or in the spiking regimes. We note again that the number of spikes per cycle strongly depends on the value of  $\phi$ .



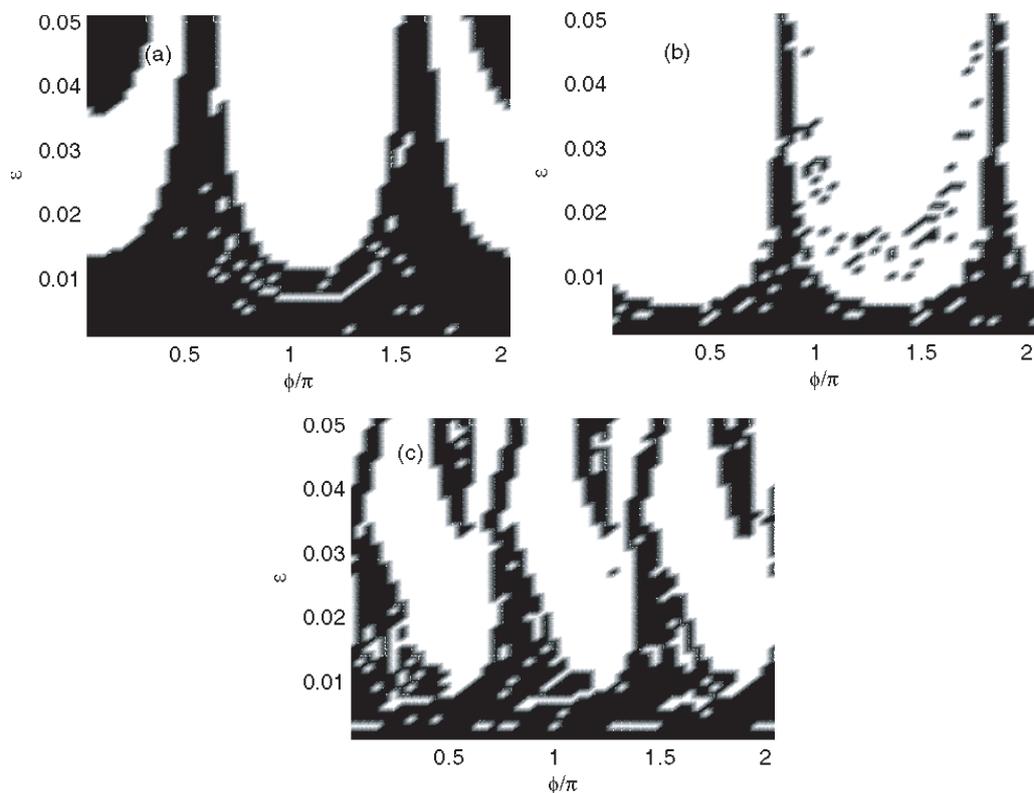
**Figure 5.** Number of spikes  $H$  per cycle obtained by applying the phase control to the FHN in the non-spiking regime, as a function of  $\epsilon$  and  $\phi$  for (a)  $r = 1$ , (b)  $r = 1/2$  and (c)  $r = 1/3$ .

Finally, we can observe in figure 6 the Lyapunov exponents obtained by applying the phase control in the non-spiking regime. We clearly observe that, again, an adequate selection of  $\phi$ , for different values of  $\epsilon$ , can tame chaotic behaviour and regularize the dynamics of the system. By combining these results with the results observed in figure 5, we find that by suitable selections of  $\epsilon$  and  $\phi$  we control the dynamics of the system in the desired state.

Summarizing, our numerical simulations show that when applying the phase control to the FHN any combination of ‘spiking’/‘non-spiking’ dynamics with ‘regular’/‘chaotic’ dynamics is possible. These numerical results are confirmed by the experiment reported in the next section.

#### 4. Experimental evidence using a circuit

In order to confirm the robustness of our control technique in the presence of noise and other distortions typical in experimental situations, we have tested the control technique in a laboratory system, an electronic circuit. As we shall see now, our results confirm the numerical findings, i.e. an adequate value of the phase difference  $\phi$  between the main driving and the controlling perturbation can control different chaotic regimes in the system. The circuit implementing the FHN model is shown in figure 7. It consists of an electronic analog simulator implemented using commercial semiconductor components.  $V_d$  is the driving voltage amplitude, applied by means of a function generator,  $V_c$  is the control voltage applied by means of another

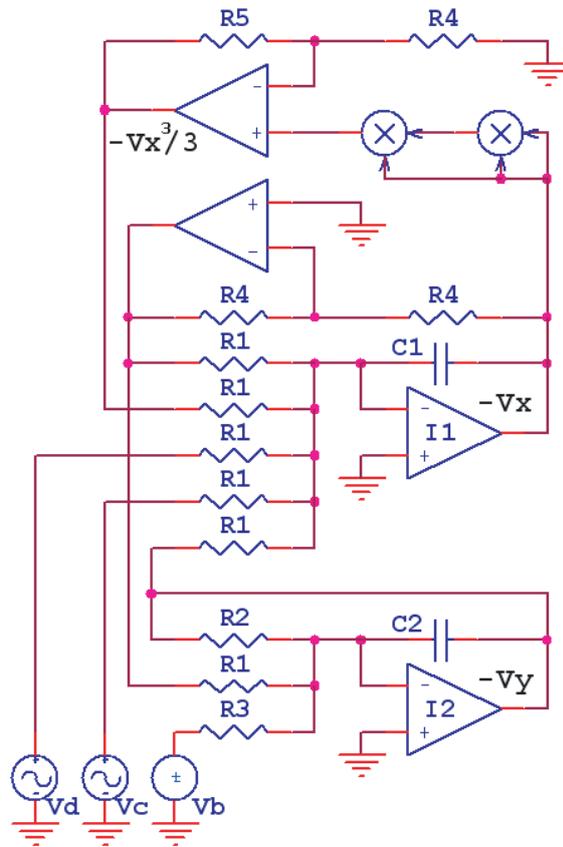


**Figure 6.** Lyapunov exponent obtained by applying the phase control to the FHN in the non-spiking regime, as a function of  $\epsilon$  and  $\phi$  for (a)  $r = 1$ , (b)  $r = 1/2$  and (c)  $r = 1/3$ . Black colour denotes positive values of LLE (chaotic regime) and white colour negative values of LLE (periodic regime).

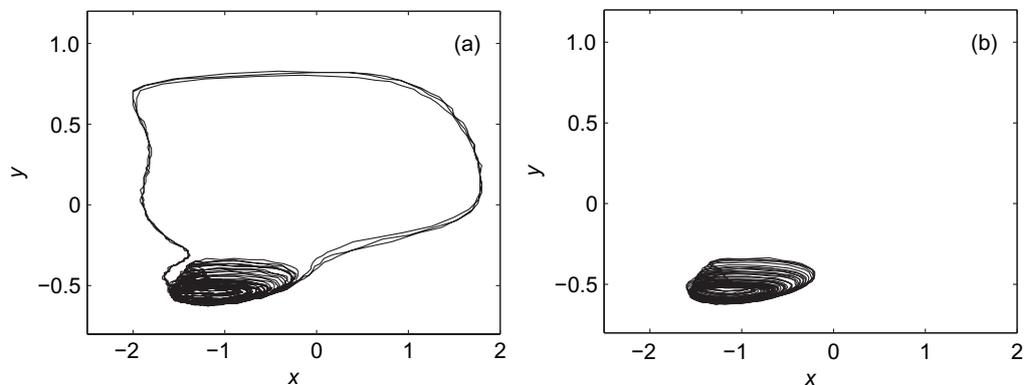
function generator and  $V_a$  is a fixed bias voltage set to  $-1$  V. The two function generators are phase-locked and their frequency ratio  $r$  can assume the values  $r = 1$ ,  $1/2$  and  $1/3$ . The integrators  $I_1$  and  $I_2$  have been implemented using Linear Technology LT1114CN quad operational amplifiers, while the four quadrant multipliers are Analog Devices MLT04. The acquisition of the experimental data has been performed by means of TEKTRONIX TDS 7104 digital oscilloscope connected to a personal computer. The voltages of the circuit  $V_x$  and  $V_y$  can be associated with  $x$  and  $y$ , respectively.

A good way to capture the influence of the phase  $\phi$  when applying the phase control scheme to a dynamical system is to perform experimental bifurcation diagrams where the perturbation amplitude  $\epsilon$  and the resonance constant  $r$  are fixed, and the phase  $\phi$  is slowly varied. These bifurcation diagrams can be built by recording very long time series where the value of the phase has been continuously varied,  $\phi(t) = \mu t$ , in such a way that the variation is much slower than the typical timescale of the dynamical system, here  $\mu \ll \omega$ . We have performed such bifurcation diagrams for different values of  $r$ , both for the chaotic non-spiking regime and the chaotic spiking regime. Experimental trajectories corresponding to these dynamical regimes are shown in figures 8(a) and (b), respectively.

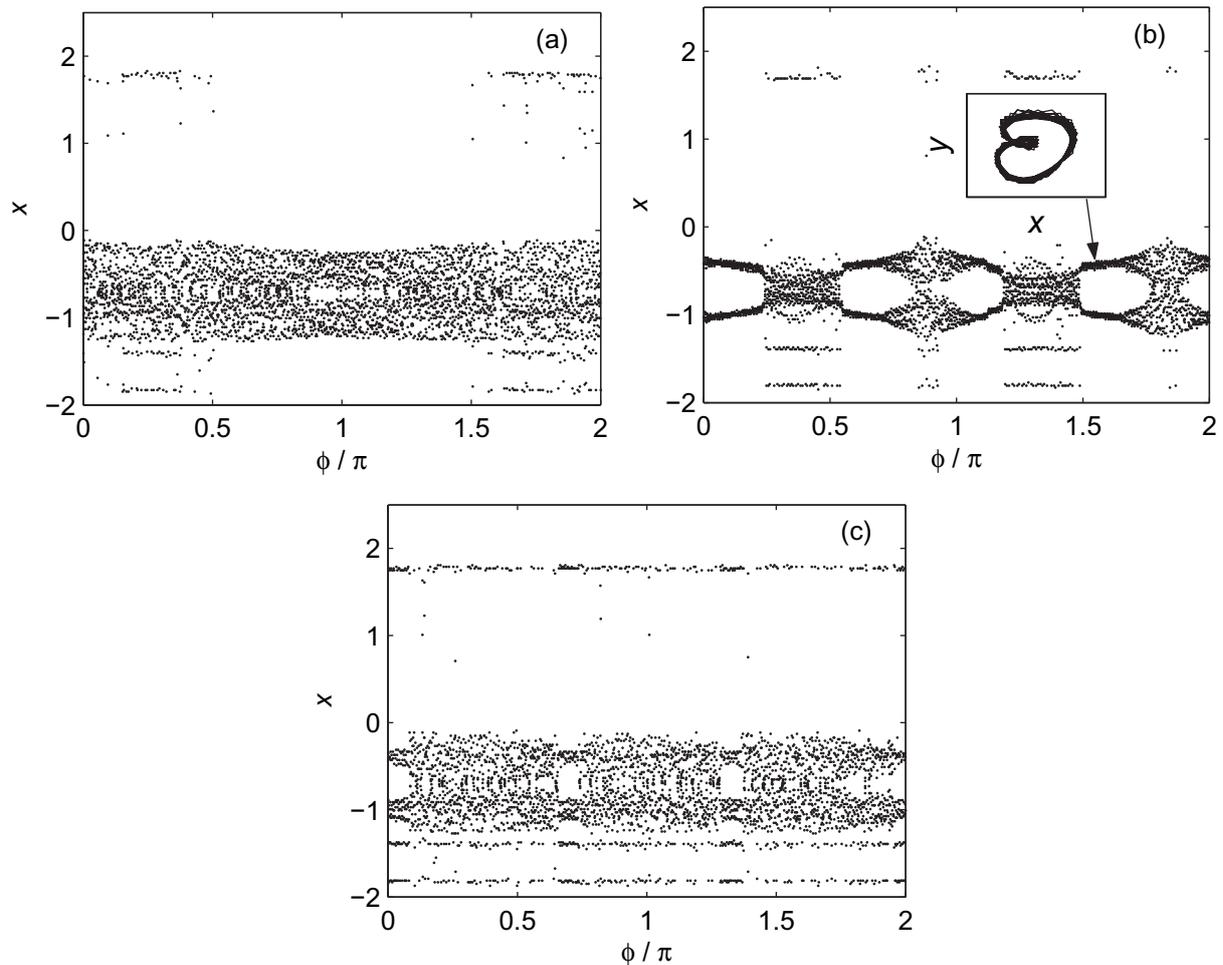
The results of applying the phase control to this circuit in the non-spiking regime are shown in figure 9. First, it is clear that in these situations the applied perturbation can induce the



**Figure 7.** Layout of the electronic circuit implementing FHN, where  $x \propto V_x$  and  $y \propto V_y$ . I, integrators; R, resistors; C, capacitors;  $\times$ , multipliers;  $V_d$ , sinusoidal driving signal;  $V_c$ , sinusoidal control signal,  $V_a$ , fixed bias voltage. The numerical values are:  $R_1 = 100 \text{ k}\Omega$ ,  $R_2 = 125 \text{ k}\Omega$ ,  $R_3 = 143 \text{ k}\Omega$ ,  $R_4 = 1 \text{ k}\Omega$ ,  $R_5 = 1.08 \text{ k}\Omega$ ,  $C_1 = 3 \text{ nF}$ ,  $C_2 = 37.5 \text{ nF}$ , and  $V_a = -1 \text{ V}$ .

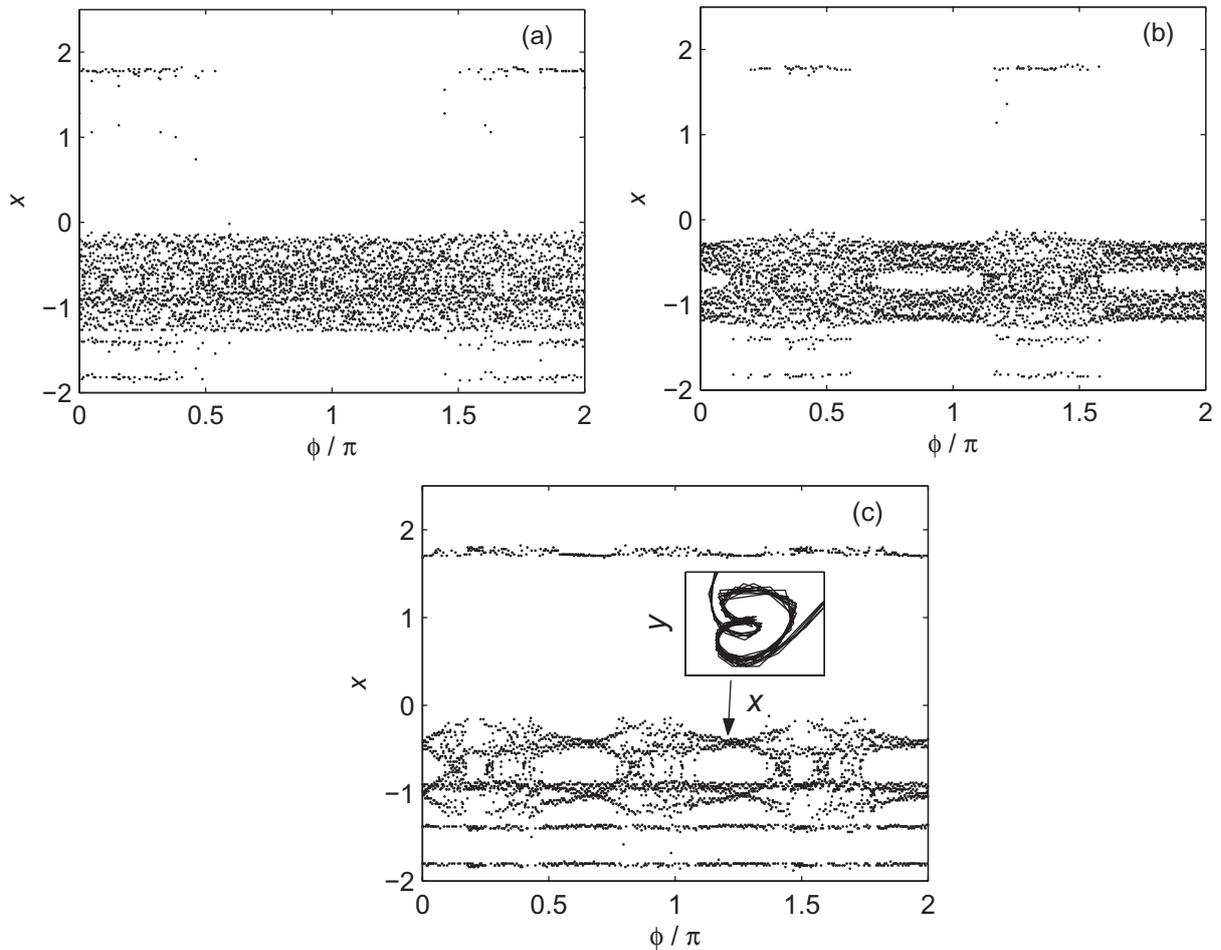


**Figure 8.** Experimental trajectories in the phase space for the periodically driven FHN model in (a) the spiking regime and (b) the non-spiking regime.



**Figure 9.** Experimental bifurcation diagrams of the trajectories when applying the phase control in the absence of spikes with (a)  $r = 1$ , (b)  $r = 1/2$  and (c)  $r = 1/3$ . The value of the phase plays a key role: it can either put the system in a spiking or in a non-spiking regime and take it from a periodic to a chaotic state. The inset plot in (b) is the plot of a trajectory obtained for the value of  $\phi$  indicated in the bifurcation diagram, that is clearly periodic and corresponds to a non-spiking regime.

appearance of spikes. The  $r = 1$  and  $r = 1/2$  cases are especially interesting, indeed figures 9(a) and (b) show that  $\phi$  plays a key role in selecting the spiking or non-spiking regimes, which is evident from the fact that for certain values of the phase  $\phi$  the bifurcation diagram has points over the threshold value. On the other hand, the phase control scheme is able to regularize the system dynamics. Furthermore, figures 9(b) and (c) confirm that an adequate value of the phase can lead the system to a periodic behaviour. The presence of environmental noise spreads the points as in small clouds, but regions of the bifurcation diagram where points are tightly clustered correspond to periodic orbits of the noisy system. This can be noticed in the inset plot of figure 9(b), where an experimental trajectory is obtained for the value of  $\phi$  indicated by the arrow is shown, that is clearly periodic and corresponds to a non-spiking regime.



**Figure 10.** Experimental bifurcation diagrams of the trajectories when applying the phase control in the spiking regime with (a)  $r = 1$ , (b)  $r = 1/2$  and (c)  $r = 1/3$ . Again, the value of the phase can tame the spikes and drive the system in a chaotic or periodic state. The inset plot in (c) is the plot of a trajectory obtained for the value of  $\phi$  indicated in the bifurcation diagram, that is clearly periodic and corresponds to the spiking regime.

Analogous phenomena can be observed when phase control is applied to the chaotic spiking regime, as shown in figure 10. In this case, we can notice that again an adequate value of  $\phi$  can tame the spikes for  $r = 1$  and  $r = 1/2$ , as shown in figures 10(a) and (b). On the other hand, the presence of clustered points in the bifurcation diagram shown in figure 10(c) is a signature of the presence of periodic behaviour for different values of  $\phi$ . This can be observed directly in the inset plot of figure 10(c), that corresponds to a trajectory for a value of  $\phi$  approximately equal to the one indicated by the arrow, clearly periodic and corresponding to a spiking regime.

## 5. Conclusions and discussions

In this work, we have provided numerical and experimental evidence that phase control can select appropriate working regimes of a FHN model. We have shown that phase control has two main effects on this model: firstly, it can tame or enhance the spiking regime and, secondly, it can turn chaotic into periodic spiking. Our work shows that phase control can be a successful approach to control the dynamics of the FHN system, and the robustness of our approach is confirmed by the experiment in a circuit. The FHN is a paradigmatic excitable system, so we expect that our control scheme might also be applied successfully to other systems of this type. On the other hand, the FHN is a good qualitative neuron model, which hints that this work might have interesting consequences in fields where control of neuronal dynamics is relevant.

## Acknowledgments

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