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Partial control of chaotic systems

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In a region in phase space where there is a chaotic saddle, all initial conditions will escape from it after a transient with the exception of a set of points of zero Lebesgue measure. The action of an external noise makes all trajectories escape faster. Attempting to avoid those escapes by applying a control smaller than noise seems to be an impossible task. Here we show, however, that this goal is indeed possible, based on a geometrical property found typically in this situation: the existence of a horseshoe. The horseshoe implies that there exist what we call safe sets, which assures that there is a general strategy that allows one to keep trajectories inside that region with control smaller than noise. We call this type of control partial control of chaos.

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INTRODUCTION

It is easy to find situations in nonlinear dynamics characterized by the presence of a nonattractive chaotic set in phase space, a chaotic saddle. Trajectories starting close to this set behave chaotically for a while, before diverging from it and settling into a periodic attractor; a phenomenon known as transient chaos. In different situations it is desirable to keep the trajectories close to the chaotic saddle, so different techniques to achieve this goal have been designed. This type of control is known as control of transient chaos [1–5], but also as chaos maintenance [6] or chaos preservation [7].

All of these control techniques face two main difficulties: the nonattractive nature of the chaotic saddle and eventually the presence of environmental noise. In these situations, the system can be described by the map $\mathbf{p}_{n+1}=\mathbf{f}(\mathbf{p}_n)$ (that can also be a Poincaré map of a flow). This map has a region Q in phase space from which nearly all trajectories escape under iterations, except those starting in the zero measure chaotic saddle (or its stable manifold). If we add noise to the system, all trajectories escape from Q. In this case, we can model the dynamics by the equation $\mathbf{p}_{n+1}=\mathbf{f}(\mathbf{p}_n)+\mathbf{u}_n$, where $\mathbf{u}_n$ is a bounded random perturbation, $\|\mathbf{u}_n\| \leq u_0 > 0$, that plays the role of *noise*. In this situation all trajectories will escape from Q under iterations, diverging thus from the chaotic saddle. A strategy to avoid those escapes is to apply an adequate *control* $\mathbf{r}_n$ to each iteration, that we assume is also bounded by a positive constant $\|\mathbf{r}_n\| \leq r_0$, in such a way that the global dynamics is given by

$$\begin{cases} \mathbf{q}_{n+1} = \mathbf{f}(\mathbf{p}_n) + \mathbf{u}_n \\ \mathbf{p}_{n+1} = \mathbf{q}_{n+1} + \mathbf{r}_n \end{cases}. \quad (1)$$

In this situation, if $r_0 > u_0$, it is not difficult to find a strategy such that trajectories can be kept inside Q, and thus close to the saddle. In order to achieve this goal with a control such that $r_0=u_0$, some of the strategies given in [1,2,4–7] can be used. However, none of these strategies allow one to keep trajectories inside Q if the control is smaller than the noise, that is, if $r_0 < u_0$. Only [3] gives a strategy that achieves this goal with $r_0 < u_0$, but it is only applicable for unimodal one-dimensional maps.

The aim of this paper is to show that in a wide variety of situations it is possible to keep trajectories close to the chaotic saddle with $r_0 < u_0$. To do this, we make explicit use of the insight that chaotic saddles are often due to the existence of a horseshoe map acting on the region Q. We are going to show that this particular geometrical action implies that there is a particular set of points inside Q, the *safe set*, with an interesting structure that we use here to design an advantageous control strategy by which trajectories can be kept inside Q if $r_0 < u_0$. Our control technique, though, does not determine exactly where the trajectory will go in Q. Thus, we call this type of control *partial control* of the system.

We want to emphasize that the existence of a horseshoe is a common situation by the celebrated Smale-Birkhoff homoclinic theorem [8,9]. This theorem states that if a map has a transverse homoclinic point, then there is a (topological) square Q such that some iterate of the map acts like a horseshoe map, whose typical action is shown in Fig. 1, and from which the existence of a chaotic saddle can be derived. This has been found to be a common situation that arises in the dynamical systems used to model different physical phenomena [9–15]. In principle, our technique can be applied in different situations of interest. As an example of application, we show here that our strategy can be applied to a paradig-

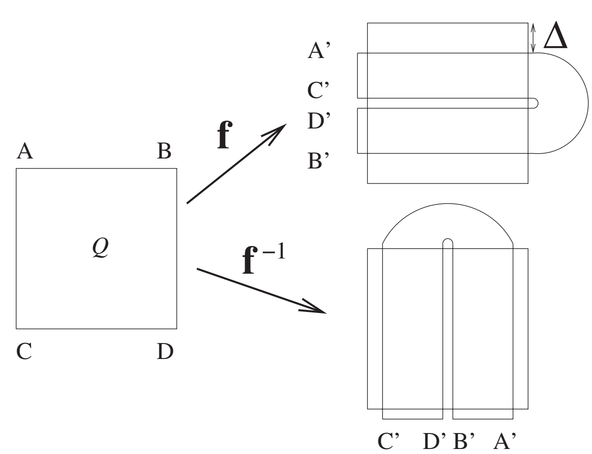

FIG. 1. The action of a horseshoe map. Δ denotes the minimum distance between the top and bottom sides of Q and $\mathbf{f}(Q)$.

matic system: a three-disk open billiard [16]. Some practical issues concerning our control method are discussed at the end of this paper.

HORSESHOE MAP

We are going to focus on how to keep trajectories close to the saddle when the system is described by Eq. (1) and $\mathbf{f}$ acts like a horseshoe map on a certain (topological) square Q. The typical geometrical action of a horseshoe on a square Q is shown in Fig. 1, which implies [8] that all trajectories escape from Q under iterations except a zero-measure set, that behaves chaotically. This is the typical situation where transient chaos arises.

We are going to use this simple model to describe our control strategy. But first, we can use it to briefly show that classical control strategies can keep trajectories inside Q only if $r_0 = u_0$. For example, an option would be to use $\mathbf{r}_n$ to steer the trajectories to points with long-lived chaotic transients (here, a Cantor set of vertical segments), as in [2]. But the presence of noise implies that trajectories will fall u_0 away from these points (i.e., if $\mathbf{p}_n$ falls in the leftmost segment), so we need $r_0 = u_0$ to make it work. Another possibility would be to try to stabilize the trajectory in one of the saddle-type periodic orbits embedded in the chaotic saddle [1]. This can be done by using $\mathbf{r}_n$ to place the trajectory of each iteration on the stable manifold (that can be locally approximated by a segment) of the saddle periodic orbit selected. But again, here the presence of noise makes this possible only if $r_0 = u_0$. Thus, these strategies would fail if $r_0 < u_0$.

With our strategy, though, we can partially control this system with $r_0 < u_0$, because for each iteration we are going to use $\mathbf{r}_n$ to steer the trajectory to the closest point of a certain set inside Q, the *safe set*, with an advantageous geometrical structure.

SAFE SETS

In general, for our partial control strategy different safe sets are needed for different values of u_0. Thus, we generate a family of safe sets $\{S^j\}$ that will allow us to partially control the system for all $u_0 > 0$. In Fig. 2 we can see how the $\{S^j\}$ are built. Consider the vertical segment that divides the square Q into two equal rectangles. We call this set of points S^0. It is easy to see that points in S^0 fall out of Q under one iteration of $\mathbf{f}$. Consider now the preimage of S^0 in Q, that we call S^1. The geometrical action of the map $\mathbf{f}^{-1}$ implies that S^1 consists of two vertical segments as shown in Fig. 2. We can now define S^2 as the preimage in Q of S^1. The geometrical action of $\mathbf{f}^{-1}$ implies that it consists of four vertical segments, as we can also see in Fig. 2. Following this procedure we can generate the set S^k for an arbitrarily high k as the preimage of S^{k-1} in Q.

Thus, we can see that each safe set $S^k \in \{S^j\}$ has the following properties:

(i) S^k consists of 2^k vertical curves.

(ii) Any vertical curve of S^k has two adjacent vertical curves of S^{k+1} *closer* to it than any other curve of S^k.

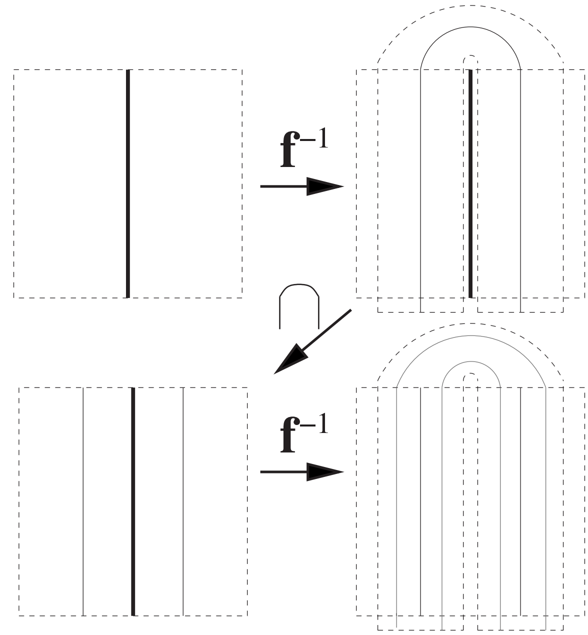

FIG. 2. The set S^0 (thick line) consists of a vertical segment in Q. Its preimage in Q consists of two vertical segments that form the safe set S^1 (black line). If we take the preimage in Q of S^1, we obtain the safe set S^2 (gray). The arrow with the label $\mathbf{f}^{-1}$ indicates that we take the inverse map, and the arrow with the label "∩" indicates that we take the intersection with the square Q. Both Q and $\mathbf{f}^{-1}(Q)$ are also shown (- -).

(iii) The maximum distance between any of the 2^k curves of S^k and its two adjacent curves of S^{k+1}, denoted as δ_k, goes to zero as $k \to \infty$.

For the horseshoe map shown in Fig. 2, the safe sets are made of vertical segments. We shall see later that for more general horseshoe maps the safe sets, built analogously, are made of vertical curves with these properties. Before we do that, we will use this horseshoe map to explain our partial control strategy.

PARTIAL CONTROL STRATEGY

Once we have generated the family of safe sets, we can describe the partial control strategy in further detail. For simplicity, we consider here that $u_0 \leq \Delta$, where Δ is the minimum distance between the top and bottom sides of Q and $\mathbf{f}(Q)$, as shown in Fig. 1 (although an analogous strategy can be implemented for $u_0 > \Delta$). Considering this, and given the value of the noise amplitude u_0, the key idea is to place the initial condition on an adequate safe set S^k. Then, we just need to apply the needed correction $\mathbf{r}_n$ to each iteration to make the point $\mathbf{p}_{n+1}$, given by Eq. (1), lie on S^k. The geometrical structure of the set S^k makes this possible even if we apply a correction that is always smaller than u_0.

The reason is the following. The adequate safe set S^k, where the initial condition $\mathbf{p}_0$ must be placed, corresponds to the smaller k value such that $\delta_{k-1} < u_0$ [which always exists no matter how small u_0 is by property (iii)]. After this, by definition $\mathbf{f}(\mathbf{p}_0)$ belongs to a curve of S^{k-1}, which has two adjacent curves of S^k. The deviation induced by noise $\mathbf{u}_0$ will

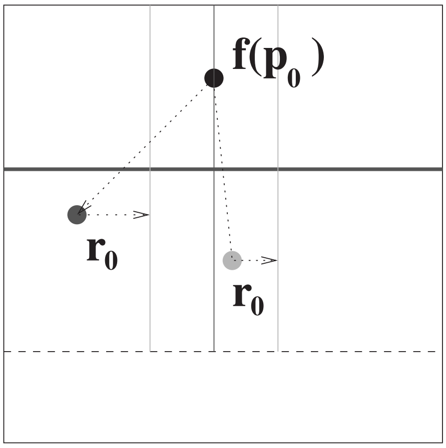

FIG. 3. The partial control strategy, illustrated by zooming on Q. The bounds of $\mathbf{f}(Q)$ (thick gray line) and of Q (dashed line) are shown for the sake of clarity. We put the initial condition $\mathbf{p}_0$ in any of the 2^k vertical curves of the safe set S^k (light gray line) such that $u_0 > \delta_{k-1}$. Then it is mapped to $\mathbf{f}(\mathbf{p}_0)$, marked with a black dot, belonging to a curve of S^{k-1} (gray line). The noise action can either deviate it to a point between the two adjacent vertical curves of S^k (gray dot), or outside this region (light gray dot). In both cases the trajectory can be placed again on the safe set S^k by applying a perturbation $\|\mathbf{r}_0\| \leq r_0 < u_0$, and this can be repeated forever.

make $\mathbf{q}_1 = \mathbf{f}(\mathbf{p}_0) + \mathbf{u}_0$ lie either in the region between those two curves of S^k or outside of it. In the former case, and by definition of δ_{k-1}, a correction $\mathbf{r}_0$ smaller than or equal to δ_{k-1} (and thus smaller than u_0) will put the trajectory on a segment of S^k. In the latter case, a correction smaller than u_0 can place it back on S^k, as we can see in Fig. 3. Following this procedure, the new point of the trajectory $\mathbf{p}_1 = \mathbf{q}_1 + \mathbf{r}_0 = \mathbf{f}(\mathbf{p}_0) + \mathbf{u}_0 + \mathbf{r}_0$ will again lie on S^k. Using the same strategy the point $\mathbf{p}_2$ will again lie on S^k, and this can be repeated forever. Thus, using this strategy we can always find a positive constant r_0 such that even if $\|\mathbf{r}_n\| \leq r_0 < u_0$, the trajectory $\mathbf{p}_n$ lies always somewhere on S^k and the system is partially controlled forever.

MAPS WITH SAFE SETS

Considering the geometrical action of $\mathbf{f}$, it is clear that for any map sufficiently similar to a horseshoe map, we can generate a family of safe sets $\{S^j\}$ with the same properties as those obtained for the map illustrated in Fig. 1. In fact, the map $\mathbf{f}$ needs to fulfill two conditions: First, $\mathbf{f}^{-1}(Q) \cap Q$ must contain two vertical strips, V_1 and V_2, and $\mathbf{f}(Q) \cap Q$ must contain two horizontal strips, H_1 and H_2, that are mapped among themselves in a horseshoelike way $\mathbf{f}(V_i) = H_i$, $\mathbf{f}^{-1}(H_i) = V_i$, $i = 1, 2$ (in the way specified by the Conley-Moser conditions [17]). Second, it needs to fulfill $\Delta > 0$, where Δ is the minimum distance between the two horizontal strips H_1 and H_2 and the top and bottom sides of Q. Under these conditions, taking as S^0 a vertical curve lying between the two vertical strips V_1 and V_2, the sets $\{S^j\}$ generated inductively as

$$S^k = \mathbf{f}^{-1}[S^{k-1} \cap (H_1 \cup H_2)] \quad (2)$$

fulfill properties (i)–(iii), so trajectories can be partially controlled here even if $r_0 < u_0$ following our strategy.

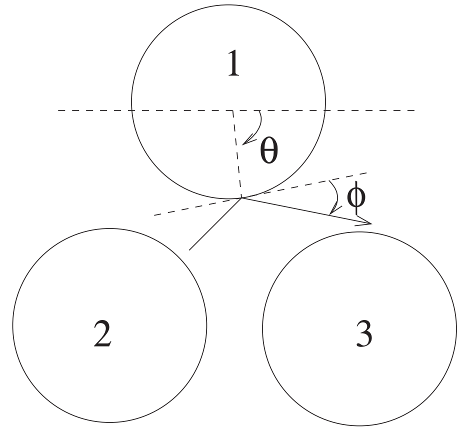

FIG. 4. The three-disk scattering problem. There is a one-to-one relation between the angles of a collision (θ_n, ϕ_n) against disk 1 and those of the next, if any.

CONTROL OF AN OPEN BILLIARD

As an example of the application of our partial control technique, we are going to show that it can be applied to the three-disk open billiard (Fig. 4). This paradigmatic chaotic scattering system [16] consists of three disks separated by a distance d, that we set $d = 2/9$. It is clear that for this system nearly all of the trajectories diverge to infinity, and a natural aim here would be to avoid such divergences using the partial control strategy.

To do this, note that each bounce against disk 1 is characterized by two angles $(\theta_n, \phi_n) \equiv \mathbf{p}_n$. In fact, there is a one-to-one relation between a bounce and the next, if any, that can be written as $\mathbf{p}_{n+1} = \mathbf{f}(\mathbf{p}_n)$. We model the presence of noise by adding a perturbation $\mathbf{u}_n$ each time that there is one such bounce, after which we apply the control $\mathbf{r}_n$ in order to keep trajectories partially controlled. Thus, this problem can be modeled by an equation analogous to Eq. (1).

In Fig. 5(a) we can see the action of the map $\mathbf{f}$ on a (topological) square Q. It is sufficiently similar to a horseshoe map in the sense specified before so we can build the safe sets $\{S^j\}$ using Eq. (2). The sets S^0, S^1, and S^2 are shown in Fig. 5(a), and they have the expected structure so that we apply our partial control strategy. As an example, a partially controlled trajectory on S^2 for $u_0 = 0.05\pi$ is shown in Fig. 5(b), and the control applied to each bounce against ball 1 is shown in Fig. 5(c), which is clearly smaller than $u_0 = 0.05\pi$.

PRACTICAL ISSUES

The detection of safe sets is an important issue. To do this, a two-step procedure is needed: First, detect Q such that $\mathbf{f}(Q)$ acts like a horseshoe map. After this, locate S^0 and, using either the explicit form of $\mathbf{f}$ when known or a (sufficiently big) number of (conveniently denoised) time series, approximate the preimages needed to use Eq. (2) and compute the safe sets. To locate the square Q, we can use two strategies that will be detailed elsewhere: First, we need to experimentally detect a saddle periodic orbit, experimentally approximate [18] its stable and unstable manifolds, and try to reproduce the Smale-Birkhoff theorem picture. The other option is

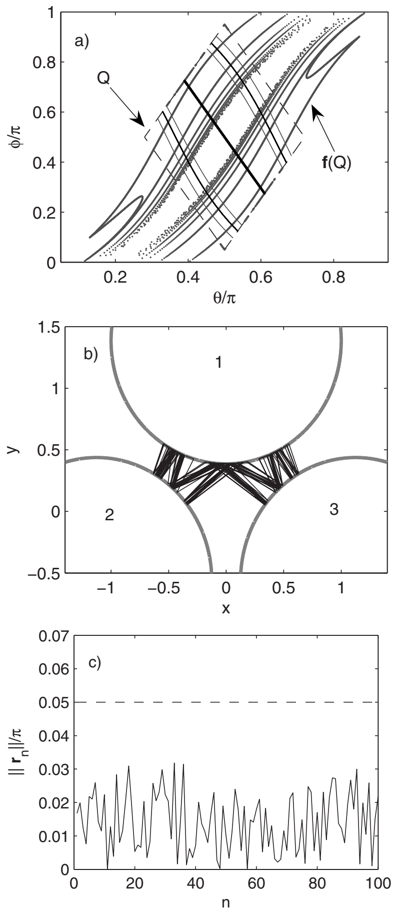

FIG. 5. Square Q (dashed), its image under $\mathbf{f}$ (gray), and the sets S^0 (thick black), S^1 (black), and S^2 (gray) for the three-disk scattering problem (a). A partially controlled trajectory when $u_0=0.05\pi$ (b) and the control applied $\|\mathbf{r}_n\|$ each bounce against the disk 1 (c), which clearly verifies $\|\mathbf{r}_n\|<u_0$ (dashed line) as expected.

to detect the chaotic saddle (using, i.e., the method proposed in [19]) and use time series to approximate a square Q enclosing it such that $\mathbf{f}(Q)$ acts like a horseshoe.

In any case (especially in experimental situations) safe sets will only be detected with limited accuracy. On the other hand, typically there will be some noise in the control applied $\mathbf{r}_n$ each iteration. From our point of view, these two situations are somehow equivalent and they can be modeled by considering that for each iteration we apply a control $\mathbf{r}'_n+\Delta\mathbf{r}_n$, where $\|\Delta\mathbf{r}_n\|<\Delta r_0$ is a random perturbation that plays the role of "control noise." But in this situation it is still possible to keep trajectories bounded even when the control applied is smaller than noise (provided that Δr_0 is small). Assume that for a given u_0, for $\Delta r_0=0$, trajectories can be partially controlled with $r_0<u_0$ on the safe set S^k. Assume that our trajectory starts in $\mathbf{p}_n$, a point that is at most Δr_0 away from the adequate safe set S^k. Due to the noise action and to the error in the control, $\mathbf{q}_{n+1}=\mathbf{f}(\mathbf{p}_n)+\mathbf{u}_n$ is at most $u_0+C\Delta r_0$ away from S^{k-1}, where $C>1$ is a constant that depends on the map $\mathbf{f}$. Using a strategy analogous to the one illustrated by Fig. 3, we can see that an accurate correction $\mathbf{r}'_n$ such that $\|\mathbf{r}'_n\|\leq r'_0\equiv r_0+C\Delta r_0$ will be enough to place the new point of the trajectory $\mathbf{p}_{n+1}=\mathbf{q}_{n+1}+\mathbf{r}'_n+\Delta\mathbf{r}_n$ at most Δr_0 away from the adequate safe set, and this can be repeated forever. If Δr_0 is sufficiently small, r'_0 is smaller than u_0, as claimed.

CONCLUSIONS

In this paper we have outlined a technique to partially control a chaotic system. This technique allows one to keep trajectories in a region of the phase space containing a chaotic saddle, even if the control applied is smaller than the noise amplitude. The main reason for this counterintuitive situation is the existence of a geometrical structure, the safe sets, which are used to keep trajectories inside the prescribed region.

ACKNOWLEDGMENTS

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