Partial Control of a System with Fractal Basin Boundaries

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Abstract. In this paper we apply the technique of partial control of a chaotic system to a dynamical system with two attractors with fractal basin boundaries, in presence of environmental noise. This technique allows one to keep the trajectories far from any of the attractors by applying a control that is smaller than the amplitude of environmental noise. We will show that the same geometrical horseshoe-like action that gives rise to the existence of fractal basin boundaries will allow us to detect certain sets for the dynamical system considered, the safe sets, that make this type of control possible.

Keywords: Fractal basin boundaries, control, transient chaos.
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INTRODUCTION

Some dynamical systems are not chaotic but they have an nonattractive invariant set that induces transient chaos [2], usually referred to as a chaotic saddle. There are different mechanisms by which a chaotic attractor is destroyed, giving rise to a chaotic saddle, for example the boundary crisis [5]. Thus, transient chaos can be found in a large variety of contexts [4] so in many different situations it might be desirable to control the system to keep its trajectories close to the chaotic saddle, or simply more convenient than letting them fall in the coexisting attractor. This idea can be framed in the wide field of control of chaotic dynamical systems [16], that has been recently found to have interesting applications in different fields of science and engineering [3].

Considering this, different techniques have been proposed in recent years to control transient chaos. Some of them are inspired in the OGY chaos control scheme [11], based on stabilization of the system around one of the unstable periodic orbits that lie in the chaotic saddle [17, 12, 13] whereas others are focused on accurately perturbing the system in order to avoid "dangerous" regions of the phase space, from which trajectories are expelled to the attractor [14, 6, 4, 1, 18].

There are two main difficulties involving this control task. The first one is the nonattracting nature of the chaotic saddle, from which trajectories typically diverge after a finite amount of time. The second one is that most dynamical systems of interest in practical applications are affected by the presence of noise. We could assume, as an extra difficulty, that our action on the system is bounded to be smaller than the action of noise. In that situation it would seem that it is impossible to sustain transient chaos. However, in a recent paper Aguirre et al [1] showed that this was indeed possible for the simplest dynamical system with a chaotic saddle and escapes to infinity: the slope-three tent map. We recently extended these ideas [19] to the family of dynamical systems for

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which a horseshoe-like map [15] is the responsible of the appearance of the chaotic saddle. We showed that for these systems it was possible to find inside a square Q enclosing the chaotic saddle (where the system considered acts as a horseshoe map) certain *safe sets* that allowed one to achieve this goal. This type of control does not say where the trajectories will exactly go in the vicinity of the chaotic saddle, so we called it *partial control* of a dynamical system.

Chaotic saddles can appear also in systems for which there is more than one attractor. Furthermore, it is quite general to find transient chaos in systems with fractal basin boundaries [7]. For these systems, the dimension of the set that separates the basins of attractions of two or more attractors has a noninteger dimension, which implies a degree of unpredictability which is related with this (fractal) dimension. This phenomenon has been related with the existence of horseshoe like-mapping in a given zones of phase space [7]. Sometimes this relation is found indirectly. For example, by using Melnikov's method [8], it was found that for certain oscillators the appearance of fractal basin boundaries is simultaneous to that of transverse homoclinic points [9] (which imply the existence of a horseshoe on phase space). Considering that our partial control technique applies for dynamical systems that present a horseshoe in phase space, we expect that our technique can be applied in this context to keep the trajectories far from any of the attractors. In fact, in this paper we are going to show that for a paradigmatic map with fractal basin boundaries it is possible to find safe sets analogous to those used in Ref. [19] that can be used to keep the trajectories of the dynamical system far from any of the periodic attractors of the system, even if the control applied is smaller than the amplitude of noise. This is done by making use of the fact that the dynamical system considered behaves approximately like a horseshoe map on a square Q of phase space. Using our technique, we will show that trajectories can be kept inside that square even if the control applied is smaller than the amplitude of noise.

The structure of this paper is the following. First we describe the dynamical system that we are going to use as an example. After this, we are going to state the control problem that we deal with in this paper in a mathematically precise way. After doing this, we will show how the safe sets can be built for this dynamical system, and we will explain why they have the structure that allows one to keep the trajectories far from the attractors using a control smaller than the amplitude of noise. Finally we are going to give a numerical example of application of our technique and we will draw the main conclusions of this work.

DESCRIPTION OF THE SYSTEM AND PROBLEM STATEMENT

The system that we consider here is a two-dimensional map $\mathbf{f}$ that has been thoroughly described in Ref. [7]. The equations of this system are

$$(\theta_{n+1}, x_{n+1}) = \mathbf{f}(\theta_n, x_n) = (\theta_n + a\sin 2\theta_n - b\sin 4\theta_n - x_n \sin\theta_n, -J\cos\theta_n), \qquad (1)$$

where the angles θ and $\theta + 2\pi$ are identified. In the following discussion, we will label the points of the trajectories of this dynamical system as $\mathbf{p}_n = (\theta_n, x_n)$, so the relation above can be written as $\mathbf{p}_{n+1} = \mathbf{f}(\mathbf{p}_n)$.

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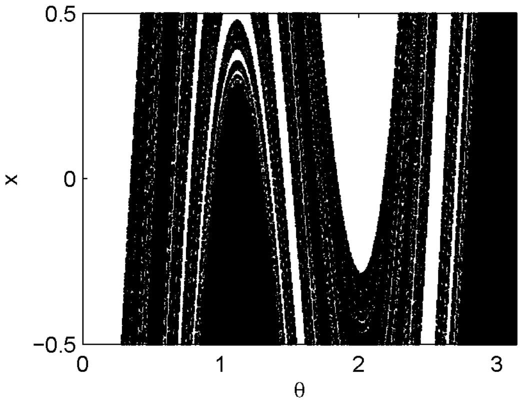

FIGURE 1. Basins of attraction of the fixed points $\mathbf{p}^- = (0,-0.3)$ (white) and $\mathbf{p}^+ = (\pi,0.3)$ (black). The existence of fractal basin boundaries is clear

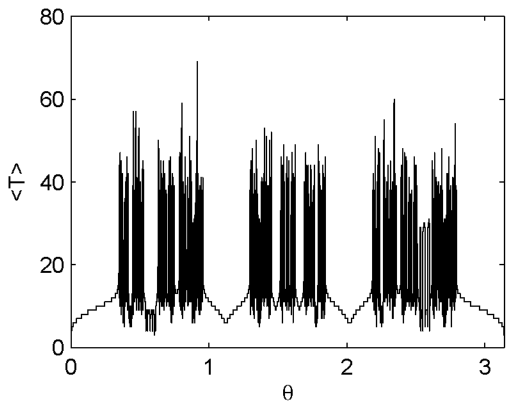

FIGURE 2. Average time $< T >$ needed by a trajectory starting in $x = 0$ and different values of θ to settle into any of the two coexisting periodic attractors

For the values of the parameters that we will use from now on, $a = 1.32$, $b = 0.9$ and $J = 0.3$ the system has two attractive fixed points, $\mathbf{p}^- = (0,-J)$ and $\mathbf{p}^+ = (\pi,J)$. For this values of the parameters, according to Ref. [7] we expect to find fractal boundaries between the basins of attraction of these systems. The basins of attraction of these attractors have been numerically computed and are shown in In Figure 1, where the points that fall after iterations in $\mathbf{p}^-$ are marked in white, whereas those that fall after iterations in $\mathbf{p}^+$ are marked in black. We can notice that the boundaries between the two basins are not smooth, they are fractal. From this figure it is clear that if the initial condition of a trajectory lies in certain regions of the θx plane, its position should be known with a high precision in order to predict whether it will settle into $\mathbf{p}^-$ or $\mathbf{p}^+$.

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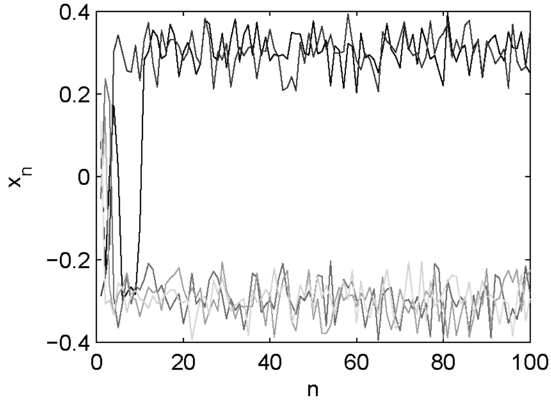

FIGURE 3. Evolution of the x coordinate of a five different trajectories of the map considered affected by noise with $u_0 = 0.1$. Note that after a finite amount of times, the x_n oscillate in the vicinity of the x coordinate of any of the two attractors of the system

The fractal boundary is a zero-measure unstable object, in the sense that trajectories are repelled from it (as long as trajectories that do not start exactly on the boundary will eventually fall in any of the attractors), and it is fractal. These are the typical features of a chaotic saddle. In fact, this system presents a behavior that is typical for systems presenting a chaotic saddle in phase space: the existence of transient chaos. In Figure 2 we represent the average time $< T >$ that a trajectory starting in $\mathbf{p}_0 = (\theta, 0)$, with θ going from 0 to π, needs to fall in the vicinity of $\mathbf{p}^-$ or $\mathbf{p}^+$. In other words, this figure shows the length of the transient needed for a trajectory to settle into a periodic orbit. It is clear from the figure that those average times depend strongly on the initial conditions, and that they can be long. This is a common feature of transient chaos.

Considering all this, what is the typical behavior of a trajectory in this situation? Typically, nearly all the trajectories (except a zero measure set) will fall arbitrarily close to either $\mathbf{p}^-$ or $\mathbf{p}^+$ after a number of iterations. Thus, the goal of our control scheme here will be to keep the trajectories far from these attractors.

As we said in the introduction, the instability of the chaotic saddle is not the only obstacle that we consider when trying to keep trajectories far from the attractor. We must not forget that most dynamical systems found in nature are under the effects of environmental noise. This situation can be modeled by adding a random perturbation to the dynamics of the system considered:

$$\mathbf{p}_{n+1} = \mathbf{f}(\mathbf{p}_n) + \mathbf{u}_n, \qquad (2)$$

where for simplicity we will assume that $\mathbf{u}_n$ is bounded by a positive constant u_0, in such a way that $||\mathbf{u}_n|| \leq u_0$. The presence of noise in this case can modify the system's dynamics, but not in a substantial way. For moderate noise values, the behavior is quite similar to the behavior described previously: the trajectories will typically wander around for a while before falling in the vicinity of the periodic points. This behavior can be observed for different trajectories in Figure 3. Depending on the initial condition

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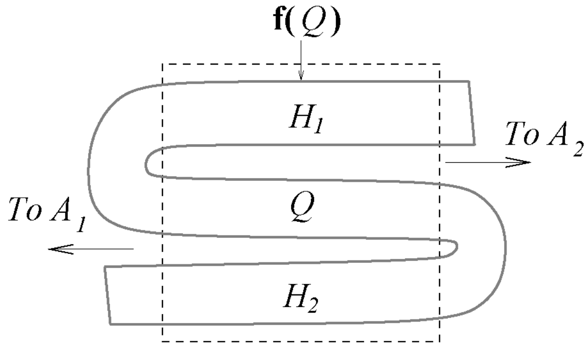

FIGURE 4. Scheme of a map for which fractal basin boundaries arise. We consider that A_1 and A_2 are attractors, in such a way that all points to the left of the square Q fall in A_1 after iterations, and all points to the right of Q fall in A_2 after iterations. Using a typical horseshoe analysis, it can be inferred that the boundaries between the basins of attraction of A_1 and A_2 are fractal. This suggests that the existence of fractal basin boundaries is related with the existence of a horseshoe-like mapping on a square Q of phase space

and on the realization of noise, the system will typically settle into any of the periodic attractors after a transient.

Thus, a natural aim here would be to preserve this transient-like behavior, i. e., to find a way to avoid the settlement into one of the periodic attractors. To do this, we can assume that we can apply each iteration an accurately chosen perturbation $\mathbf{r}_n$ to avoid this phenomenon, so the dynamics of the system considered would be given by the following equations:

$$\begin{cases} \mathbf{q}_{n+1} = \mathbf{f}(\mathbf{p}_n) + \mathbf{u}_n \\ \mathbf{p}_{n+1} = \mathbf{q}_{n+1} + \mathbf{r}_n. \end{cases} \tag{3}$$

We can assume that, as it often occurs when a system needs to be controlled, the accurately chosen perturbations $\mathbf{r_n}$ are bounded by a positive constant r_0, so $||\mathbf{r_n}|| \leq r_0$. In next section we are going to show that for this system it is possible to find the safe sets that will allow us to keep the trajectories far from the attractors even if $r_0 < u_0$.

SAFE SETS AND THE CONTROL STRATEGY

We can say now in a more precise way why we expect to find those safe sets for this dynamical system. In Ref. [19] we showed that for any dynamical system whose geometrical action reminds to that of the horseshoe map on a given square Q (and thus from which nearly all the trajectories escape) it was possible to find certain sets, the safe sets, that allowed to keep the trajectories inside Q with $r_0 < u_0$. This applies for any map that stretches and folds a number of times a given square Q in phase space. From a more technical point of view, these safe sets can be found for any map $\mathbf{f}$ that satisfies the Conley-Moser conditions [10].

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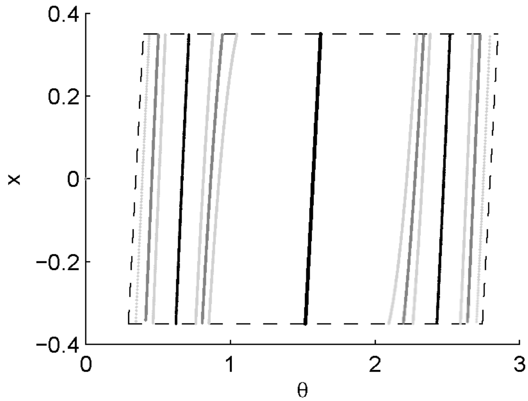

FIGURE 5. Safe sets S^k for the map considered inside the topological square Q (dashed): $k = 0$ (thick black), $k = 1$ (black), $k = 2$ (gray), $k = 3$ (light gray)

As we said in the beginning, the existence of a chaotic saddle in phase space is typically due to the existence of a horseshoe map, so in this particular case in which we have a fractal chaotic set (the fractal basin boundaries) we might find some type of horseshoe of phase space. This intuition is supported by the reasoning made in Ref. [7] by which the typical action of a map giving rise to fractal basin boundaries should be like the one shown in Figure 4.

We can explore this idea in further detail. In that figure we can see the action of a map $\mathbf{f}$ on a square Q of phase space. The map stretches and folds Q in a horseshoe-like way, so all trajectories diverge from Q. On the other hand, we assume that trajectories starting to the left of Q settle into the attractor A_1 after iterations of the map, whereas those starting to the right of the square settle into the attractor A_2. By following a reasoning that reminds to the one used in the construction of the invariant set for a Smale horseshoe map [2], it can be shown [7] that the basins of attraction of A_1 and A_2 inside Q will have an intricate appearance, and that the boundary between these sets will be fractal.

Thus, we expect to be able to find the safe sets that can be found for horseshoe-like maps in a system with fractal basin boundaries due to its underlying geometrical action. In this case, we have that those safe sets $S^k \in \{S^j\}$ would lie in a (topological) square Q between the two attractors of the system considered, $\mathbf{p}^+$ and $\mathbf{p}^-$, i.e., where the map acts like a horseshoe. We expect them to satisfy the following properties [19]:

(i) S^k is part of the preimage of S^{k-1}, and it consists of 2^k vertical curves (from top to bottom of Q).

(ii) Any vertical curve of S^k has two adjacent vertical curves of S^{k+1} *closer* to it than any other curve of S^k.

(iii) The maximum distance between any of the 2^k curves of S^k and its two adjacent curves of S^{k+1}, denoted as δ_k, goes to zero as $k \to \infty$.

(iv) For any point $\mathbf{p} \in S^k$, the distance between $\mathbf{f}(\mathbf{p})$ and the top and bottom sides of Q is $\Delta > 0$.

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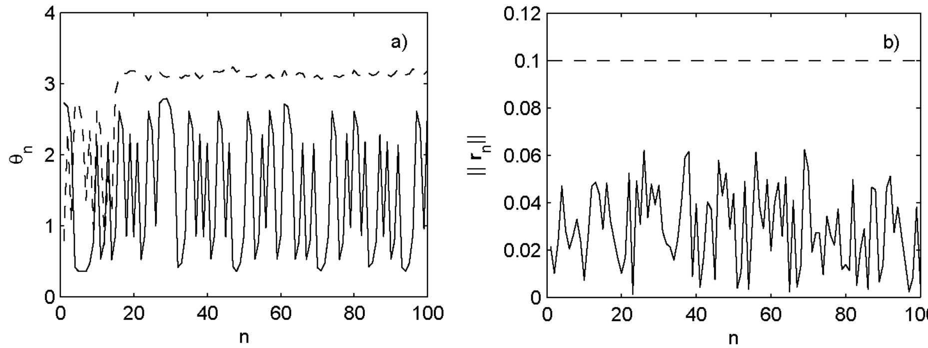

FIGURE 6. (a) Evolution of the θ variable of the system considered when the system is controlled (solid) and under noise (dashed) for $u_0 = 0.1$. The controlled trajectory, contrarily to what happened in absence of control, is kept far from the attractors. (b) Applied control for this time series (solid), which is obviously smaller than the value u_0 (marked with a dashed line)

In fact, safe sets can be found without necessarily building the horseshoe map explicitly, following a procedure that we will detail elsewhere. Those safe sets are shown here up to $k = 3$ in Figure 5. It is easy to see that they satisfy properties $(i) - (iii)$ (property (iv) can be verified mathematically in an easy way, considering that by equation 1 we have that the x variable will always be between $-J$ and J under iterations of the map). In next section we are going to describe briefly how these safe sets can be used as the key ingredient of our partial control strategy and we are going to show a numerical example of application of this technique.

THE CONTROL STRATEGY. EXAMPLE OF APPLICATION

We outline now the strategy that allows to keep the trajectory inside the square Q once the safe sets $\{S^j\}$ have been found, although a more detailed description can be found in Ref. [19]. For simplicity we assume here that u_0 is smaller than the minimum distance Δ described in property (iv) of the safe sets. Given u_0, we have to find the set S^k such that $\delta_k < u_0$, which is always possible by property (iii). Then, put the initial condition $\mathbf{p}$ in any point on S^k. The action of the map will take the trajectory to $\mathbf{f}(\mathbf{p})$, that by definition will lie in one of the 2^{k-1} curves of S^{k-1}. After this, the noise acts. But it is not difficult to see that the fact that any curve of S^{k-1} is surrounded by two adjacent curves of S^k (property (ii)) allows one to use a correction $||\mathbf{r}|| < u_0$ that puts the resulting point $\mathbf{f}(\mathbf{p}) + \mathbf{u} + \mathbf{r}$ back on a point of S^k, and this can be repeated forever. Note that this implies that we can find a value of the control r_0 such that trajectories can be kept inside Q even if $r_0 < u_0$ for any $u_0 > 0$.

As an example, we are going to control here a trajectory of the system affected by noise such that $u_0 = 0.1$. A numerical calculation shows that the adequate safe set where

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the trajectories can be stabilized is S^3. Thus, we just have to put the initial condition on any of the points of S^3 and apply each iteration the minimum correction $\mathbf{r}_n$ that allows to put the trajectory back on S^3.

A numerical example of application of our control technique is shown in Figure 6. We can see in Figure 6 (a) a controlled trajectory (solid) plotted together to an uncontrolled one (dashed). We can see that the controlled trajectory is kept far from the attractor, whereas the uncontrolled one oscillates around the fixed point. On the other hand, Figure 6 (b) shows clearly the main feature of our partial control scheme: that the amplitude of the control applied each iteration $||\mathbf{r}_n||$ is always smaller than u_0, as claimed although the trajectory follows an erratic chaotic-like behavior.

CONCLUSIONS

In this paper we have shown that the technique of partial control of a chaotic system, that typically applies to dynamical systems similar to a horseshoe map, can be applied to a paradigmatic dynamical system presenting fractal basin boundaries. We have shown that the underlying geometrical action of the system considered, that is responsible for the existence of those fractal boundaries, is related to that of a horseshoe map, and thus the safe sets necessary for our partial control strategy can be found. The main consequence of this is that trajectories can be kept far from any of the attractors even in presence of noise. Furthermore, the applied control is smaller than the action of noise. We speculate that an analogous control strategy can be applied to avoid the periodic attractors of other dynamical systems with fractal basin boundaries.

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