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EFFECT OF STEP SIZE ON BIFURCATIONS AND CHAOS OF A MAP-BASED BVP OSCILLATOR

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The continuous Bonhoeffer–van der Pol (BVP for short) oscillator is transformed into a map-based BVP model by using the forward Euler scheme. At first, the bifurcations and chaos of the map-based BVP model are investigated when the step size varies as a bifurcation parameter. By using the fast-slow decomposition technique, a two-parameter bifurcation diagram is obtained to give insight into the effect of the step size on bifurcations and chaos of the map-based BVP model. The investigation shows that the period-doubling bifurcation is dependent on the step size, while the saddle-node bifurcation is independent of the step size. Second, when the fast–slow decomposition technique cannot be used, we rigorously prove that in the map-based BVP model there exists chaos in the sense of Marotto when the discrete step size varies as a bifurcation parameter. These results show that the discrete step sizes play a vital role between the continuous-time dynamical system and the corresponding discrete dynamical system. Much attention should be paid on the step size when a map-based neuron model is used as an alternative to a continuous neuron model.

Keywords: Step size; Bonhoeffer–van der Pol oscillators; bifurcation; chaos; map-based BVP model.

1. Introduction

Map-based neuron models have received much attention over the past decade, especially in the large-scale numerical simulation of collective behavior of neuron networks [Rulkov *et al.*, 2004; Izhikevich, 2007]. As an advantageous simplification, map-based models have shown to be comparable to continuous neuron models in reproducing characteristic behavior of biological neurons [Rulkov, 2001; De Vries, 2001; Shilnikov & Rulkov, 2004; Casado, 2003; Casado *et al.*, 2004; Ibarz *et al.*, 2007a, 2007b, 2007c, 2008; Tanaka *et al.*, 2006; Cao & Sanjuan, 2009].

In general, a map-based neuron model can be obtained in two ways: one is obtained by using the discrete processing for an ODE such as the Euler discrete scheme [Rulkov *et al.*, 2004; Izhikevich, 2007], in which the step size is used to make the partition of the continuous time; the other one is obtained by using the Poincaré map [Terman, 2005].

Although it is easier to get a map-based neuron model by using the Euler discrete method than by using the Poincaré map, many problems remain unclear. The basic question is whether the qualitative properties of continuous-time systems are preserved via discretization methods. Another one of the very important problems is the effect of the step size on the nonlinear dynamical behaviors of a map-based neuron model. Concerning these questions, as a kind of special Morse–Smale dynamical system [Shub, 2007], the continuous-time gradient system on a two-dimensional compact manifold is globally topologically conjugate to the corresponding discrete dynamical system obtained by Euler method for a sufficiently small time step [Bielecki, 2002]. In addition, it is noted that the similar question has been considered by other authors [Jing *et al.*, 2002]. While, there is still less attention on non-Morse–Smale dynamical system, especially when some nonhyperbolic continuous dynamical systems (i.e. structurally unstable systems) are concerned.

In this paper, as a simplified version to the Hodgin–Huxley nerve equations [Hodgkin & Huxley, 1952], the continuous BVP oscillator is transformed into a map-based BVP model by using the forward Euler scheme. As a representative of a wide class of nonlinear excitable oscillator, BVP oscillator is taken into account not only because BVP oscillator has wider application in the modeling of biological precesses, but also has rich nonlinear behavior including different topological properties between the continuous-time BVP oscillator and the discrete one. Therefore, our main goal in this paper is to investigate the effect of the step size as a bifurcation parameter on bifurcations and chaos of a map-based BVP model. At first, the fast–slow decomposition technique is used to analyze the fast subsystem of a map-based BVP model. A two-parameter bifurcation diagram is obtained to give insight into the effect of the step size on bifurcations and chaos of a map-based BVP model. The investigation demonstrates that the curve of fixed points, the saddle-node bifurcations, are independent of the step size, while the period-doubling bifurcation is dependent on the variation of the step size. Second, when the fast–slow decomposition technique fails, we rigorously prove that in the discrete BVP model there exists chaos in the sense of Marotto [2005] when the step size varies as a bifurcation parameter. The existence condition for chaos is different to that presented in [Jing *et al.*, 2002]. These results show that the discrete step sizes play a vital role between the continuous-time dynamical system and the corresponding discrete dynamical system. Much attention should be paid on the step size when a map-based neuron model is used as an alternative to a continuous neuron model.

The layout of this paper is as follows. In Sec. 2, a two-parameter bifurcation analysis is given by using the fast–slow decomposition technique. The strict mathematical analysis is presented concerning the effect of the step size on bifurcations and chaos in Sec. 3. Finally, we sum up our results in Sec. 4.

2. The Single Map-Based BVP Model

The classical continuous BVP oscillator may be written as

$$\begin{aligned} \dot{x} &= y - \frac{1}{3}x^3 + x + \mu, \\ \dot{y} &= \rho(a - x - by), \end{aligned} \tag{1}$$

and it can be transformed into a two-dimensional map by using the forward Euler discrete scheme

$$\begin{aligned} x_{n+1} &= x_n + \delta\left(y_n - \frac{1}{3}x_n^3 + x_n + \mu\right), \\ y_{n+1} &= y_n + \delta\rho(a - x_n - by_n), \end{aligned} \tag{2}$$

where $0 < \rho \ll 1, 0 < a < 1, 0 < b < 1, \mu$ is a stimulus intensity, and $0 < \delta < 1$ is the step size. The state variable x can be thought of the electric potential across the cell membrane, and the other state variable y stands for a recovery force. Due to the fact that $0 < \delta\rho \ll 1$ is small enough, the evolution of y (or y_n) is much slower than that of x (or x_n). Thus, we refer to x_n as the fast variable and y_n as the slow variable.

The bifurcation analysis of Eq. (2) can be obtained by using the fast–slow decomposition technique, which was at first used by [Rinzel, 1987] when a continuous bursting system was considered. Afterwards, this technique was widely used to different continuous or discrete systems by many researchers such as [Sherman, 1996; Rulkov, 2001; De Vries, 2001; Izhikevich, 2007; Casado *et al.*, 2003, 2004; Ibarz *et al.*, 2007a, 2007b, 2007c; Tanaka *et al.*, 2006; Cao & Sanjuan, 2009].

The key to the fast–slow decomposition technique is to consider the slow variable y_n as a bifurcation parameter, and to substitute it into the first equation of Eq. (2). Thus, the bifurcation analysis of Eq. (2) can be investigated through the following fast subsystem by letting $y_n + \mu$ as a bifurcation parameter γ, that is,

$$x_{n+1} = x_n - \delta\frac{1}{3}x_n^3 + \delta x_n + \delta\gamma. \tag{3}$$

Moreover Eq. (3) is symmetric after the transformation $\gamma \rightarrow -\gamma$ and $x_n \rightarrow -x_n$. Thus, the bifurcation diagram is symmetric to the origin. The two-parameter γ–δ bifurcation diagram for the fast subsystem Eq. (3) is shown in Fig. 1(a). The solid lines represent the curves of saddle-node bifurcations γ_{SN} given by the two straight lines

$$\gamma = \pm\frac{2}{3}. \tag{4}$$

The dotted lines represent the curves of period-doubling bifurcations γ_{PD} given by

$$\gamma = \pm\frac{2}{3\delta}(1-\delta)\sqrt{\frac{2+\delta}{\delta}}. \tag{5}$$

Seen from Fig. 1(a), there exist two saddle-node bifurcations and two period-doubling bifurcations corresponding to each fixed value of the parameter δ when $0 < \delta < 1$. Corresponding to each

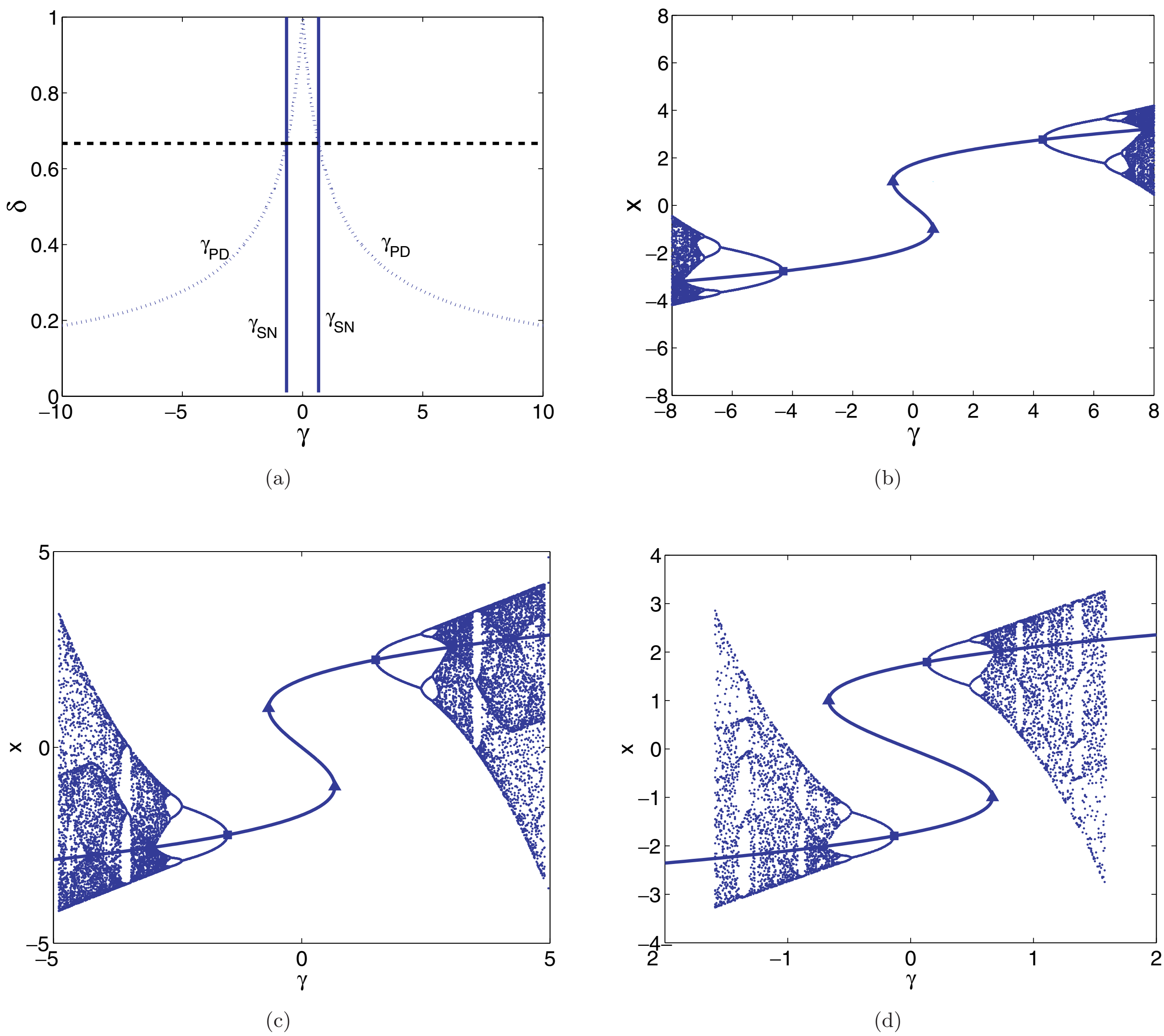

Fig. 1. (a) Two-parameter bifurcation diagram for the fast subsystem Eq. (3). (b) Bifurcation diagram for the fast subsystem with $\delta = 0.3$. (c) Bifurcation diagram for the fast subsystem with $\delta = 0.5$. (d) Bifurcation diagram for the fast subsystem with $\delta = 0.9$.

δ $(0 < \delta < 1)$, the horizontal distance between the left-hand saddle-node bifurcation and the right-hand saddle-node bifurcation is always fixed at 3/4, which is independent of the variation of δ. In contrast with the situation in the saddle-node bifurcation, with the increasing step size δ, the horizontal distance between the left-hand period-doubling bifurcation and the right-hand period-doubling bifurcation is decreasing, and finally tends to the zero distance. In addition, the bifurcation analysis shows that there does not exist the external/internal crisis bifurcation, which means that the minimum of the iterate cannot map onto an unstable fixed point on the middle branch of the curve of fixed points. This implies that bursting cannot take place resulting from bistability [De Vries, 2001].

The following three bifurcation diagrams Figs. 1(b)–1(d) further explain the bifurcation phenomenon given in Fig. 1(a) corresponding to $\delta = 0.3$, $\delta = 0.5$, and $\delta = 0.9$, respectively. Seen from Figs. 1(b)–1(d), there exists at first an S-shaped curve of fixed points. Second, there exist two saddle-node bifurcations (denoted by triangles) occurring at the knees of these S-shaped fixed point curves. Third, there exist two period-doubling bifurcations (denoted by squares). When the bifurcation parameter γ is approaching each period-doubling bifurcation, stable two-cycle, four-cycles, eight-cycles, etc., are observed. Finally, the route from period-doubling bifurcation to chaotic attractors is also observed. The strict mathematical proof will be discussed later.

3. Mathematical Analysis of the Two-Dimensional Map-Based BVP Model

If ρ is not small enough, then the fast–slow decomposition technique cannot be used again. So in this section, we will analyze the two-dimensional Map-Based BVP model.

The equivalent map form of Eq. (2) can be rewritten as

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + \delta\left(y - \frac{1}{3}x^3 + x + \mu\right) \\ y + \delta\rho(a - x - by) \end{pmatrix}. \quad (6)$$

3.1. *Fixed points*

The fixed point of Eq. (6) satisfies the following equations:

$$\begin{aligned} &y - \frac{1}{3}x^3 + x + \mu = 0, \\ &a - x - by = 0, \end{aligned} \quad (7)$$

that is,

$$-\frac{1}{3}x^3 + \left(1 - \frac{1}{b}\right)x + \frac{a}{b} + \mu = 0. \quad (8)$$

There exists only a real fixed point of F due to the discriminant $\Delta = 9((a/b)+\mu)^2 - 4(1-(1/b))^3 > 0$ when $b < 1$. And the unique real fixed point of F is presented as follows:

$$x = -\frac{-2b + 2b^2 + \sqrt[3]{2}(-3ab^2 - 3b^3\mu + \sqrt{b^3[-4(b-1)^3 + 9b(a+b\mu)^2]})^{\frac{2}{3}}}{\sqrt[3]{4}b(-3ab^2 - 3b^3\mu + \sqrt{b^3(-4(b-1)^3 + 9b(a+b\mu)^2)})^{\frac{1}{3}}}. \quad (9)$$

We have the following proposition:

Proposition 1. *For any value of a, μ, δ, if $b < 1$, then there exists only one fixed point of F, and the forward Euler discrete scheme does not make any change for the fixed point of Eq. (1) with respect to the original continuous BVP oscillator.*

3.2. *Stability analysis*

The Jacobian matrix of F at fixed points is as follows:

$$DF(x) = \begin{pmatrix} 1 + \delta - \delta x^2 & \delta \\ -\delta\rho & 1 - \delta\rho b \end{pmatrix}, \quad (10)$$

and the corresponding characteristic equation of the Jacobian matrix $DF(x)$ at fixed points can be expressed as

$$\lambda^2 + p(x)\lambda + q(x) = 0, \quad (11)$$

where

$$\begin{aligned} p(x) &= \delta x^2 + \delta\rho b - \delta - 2, \\ q(x) &= (\delta^2\rho b - \delta)x^2 - \delta^2\rho b + \delta^2\rho \\ &\quad - \delta\rho b + \delta + 1. \end{aligned} \quad (12)$$

It is noted that the fixed points of F satisfy Eq. (8), then the discriminant of Eq. (11) can be written as

$$\Delta = p^2(x) - 4q(x) = Ax^2 + Bx + c = 0, \quad (13)$$

and where

$$A = \delta^2 - \frac{3}{b}\delta^2 - 2\delta^2\rho b,$$
$$B = \frac{3\delta^2 a}{b} + 3\delta^2\mu, \quad (14)$$
$$C = \delta^2 b^2\rho^2 + 2\delta^2\rho b - 4\delta^2\rho + \delta^2.$$

It is obvious that $A < 0$ if $0 < \rho < 1, 0 < a < 1, 0 < b < 1$, and $0 < \delta < 1$. Then, the discriminant of Eq. (13) is as follows:

$$\Delta_{13} = 9a^2 + 12b - 4b^2 + 18ab\mu + 9b\mu^2 - 48b\mu + 40b^2\rho - 20b^3\rho^2 + 12b^4\rho^2 + 8b^5\rho^3. \quad (15)$$

Thus, there exist two real roots x_1 and x_2 of Eq. (13) due to $\Delta_{13} > 0$ when $1/4 < \rho < 1, (2\sqrt{\rho} - 1)/\rho < b < 1$, and $0 < \delta < 1$, where

$$x_1 = \frac{-3a - 3b\mu - \sqrt{9(a + b\mu)^2 + 4b(3 + 2b^2\rho - b)(1 + 2(b - 2)\rho) + b^2\rho^2}}{2(-3 + b - 2b^2\rho)},$$
$$x_2 = \frac{-3a - 3b\mu + \sqrt{9(a + b\mu)^2 + 4b(3 + 2b^2\rho - b)(1 + 2(b - 2)\rho) + b^2\rho^2}}{2(-3 + b - 2b^2\rho)}. \quad (16)$$

Without loss of generality, we suppose that $x_1 < x_2$, then we have the following proposition:

Proposition 2. *If* $1/4 < \rho < 1, (2\sqrt{\rho} - 1)/\rho < b < 1$, *and* $0 < \delta < 1$, *then the fixed point of* F *is unstable if one of the following conditions is satisfied*:

(i) *when* $x_1 < x < x_2$, *and*

$$|x| < \sqrt{\frac{\delta^2\rho b - \delta^2\rho + 2\delta\rho b - 2\delta - 4}{\delta^2\rho b - 2\delta}};$$

(ii) *when* $x = x_1$ *or* $x = x_2$, *and*

$$x > \sqrt{\frac{4 + \delta - \delta\rho b}{\delta^2}} \quad \textit{or} \quad x < -\sqrt{\frac{4 + \delta - \delta\rho b}{\delta^2}};$$

(iii) *when* $x < x_1$ *or* $x > x_2$, *and*

$$x > \sqrt{\frac{\delta^2\rho b - \delta^2\rho + \delta\rho b - \delta}{\delta\rho b - 1}}$$

or

$$x < -\sqrt{\frac{\delta^2\rho b - \delta^2\rho + \delta\rho b - \delta}{\delta\rho b - 1}}.$$

4. Chaos in the Sense of Marotto

A redefining snap-back repeller is given as follows [Marotto, 2005]:

Definition. Suppose z is a fixed point of f with all eigenvalues of $Df(z)$ exceeding 1 in magnitude, and suppose there exists a point $x_0 \neq z$ in a repelling neighborhood of z, such that $x_M = z$ and $\det(Df(x_k)) \neq 0$ for $1 \leq k \leq M$, where $x_k = f^k(x_0)$. Then z is called a snap-back repeller of f.

Then, the chaos in the sense of Marotto is as follows:

Theorem. *If* f *has a snap-back repeller then* f *is chaotic.*

In the following, we will prove that there exists a snap-back repeller in the sense of Marotto concerning the following simplified form of Eq. (17), where a shift transformation has been taken in order to guarantee that $(0, 0)$ is the only fixed point of the following equations:

$$x_{n+1} = x_n + \delta\left(y_n - \frac{1}{3}x_n^3 + x_n\right),$$
$$y_{n+1} = y_n + \delta\rho(-x_n - by_n). \quad (17)$$

We will at first prove the unique fixed point $O(0, 0)$ to be an expanding fixed point.

The characteristic equation of the Jacobian matrix of Eq. (17) at $O(0, 0)$ can be written as

$$H_0(\lambda) = \lambda^2 + p(0)\lambda + q(0) = 0, \quad (18)$$

where

$$p(0) = \delta\rho b - \delta - 2,$$
$$q(0) = 1 - \delta\rho b + \delta - \delta^2\rho b + \delta^2\rho. \quad (19)$$

Suppose that λ_1 and λ_2 are two characteristic roots of Eq. (25), then

$$\lambda_1 + \lambda_2 = -p(0), \quad \lambda_1\lambda_2 = q(0). \quad (20)$$

The discriminant of the characteristic equation of Eq. (18) is as follows:

$$\begin{aligned}\Delta_0 &= p^2(0) - 4q(0)\\ &= \delta^2(1 - 4\rho + 2b\rho + b^2\rho^2).\end{aligned} \tag{21}$$

We have then the following proposition:

Proposition 3

(i) *If* $1/4 < \rho < 1$, $0 < b < (2\sqrt{\rho} - 1)/\rho$, *and* $0 < \delta < 1$, *then* $\Delta_0 < 0$;
(ii) *If* $1/4 < \rho < 1$, $b = (2\sqrt{\rho} - 1)/\rho$, *and* $0 < \delta < 1$, *then* $\Delta_0 = 0$;
(iii) *If* $1/4 < \rho < 1$, $(2\sqrt{\rho} - 1)/\rho < b < 1$, *and* $0 < \delta < 1$, *then* $\Delta_0 > 0$.

By using the stability analysis at the fixed point $O(0,0)$, the following proposition can be presented:

Proposition 4

(i) *If* $\Delta_0 > 0$, *then there exist two unequal real roots* λ_1 *and* λ_2 *of Eq. (18). And if* $H_0(-1) = \delta^2\rho - \delta^2\rho b < 0$ *and* $H_0(1) = 4 - 2\delta^2\rho b + 2\delta - \delta^2\rho b + \delta^2 b < 0$, *then* $|\lambda_i| > 1$ $(i = 1, 2)$;
(ii) *If* $\Delta_0 = 0$, *then there exist two equal real roots* $\lambda_1 = \lambda_2$ *of Eq. (18). And if* $|p(0)| = |\delta\rho b - \delta - 2| < 2$, *then* $|\lambda_i| > 1$ $(i = 1, 2)$;
(iii) *If* $\Delta_0 < 0$, *then there exist a pair of pure imaginary roots* λ_1, *and* $\overline{\lambda_1}$ *of Eq. (18). And if* $|q(0)| = |1 - \delta\rho b + \delta - \delta^2\rho b + \delta^2\rho| > 1$, *then* $|\lambda_i| > 1$ $(i = 1, 2)$.

To sum up, the condition of expanding fixed point is obtained as follows:

Proposition 5. *If* $1/4 < \rho < 1, 0 < b < (2\sqrt{\rho} - 1)/\rho$, *and* $0 < \delta < 1$, *then the fixed point* $O(0,0)$ *is an expanding fixed point.*

Next, we want to find another point $Z(x, y)$ in the neighborhood of $O(0,0)$ satisfying $F^2(Z) = O, Z(x,y) \neq O(0,0)$, and $\|DF^2(Z)\| \neq 0$.

A 2-period circle satisfies the following four equations:

$$\begin{aligned}(1+\delta)x - \frac{1}{3}\delta x^3 + \delta y &= X,\\ -\delta\rho x + (1 - \delta\rho b)y &= Y,\\ (1+\delta)X - \frac{1}{3}\delta X^3 + \delta Y &= 0,\\ -\delta\rho X + (1 - \delta\rho b)Y &= 0.\end{aligned} \tag{22}$$

X, Y, and y can be obtained as follows:

$$\begin{aligned}X &= \frac{\delta(1 - \delta\rho b)^2}{\delta^2\rho - 3(1 - \delta\rho b)^2},\\ Y &= \frac{\delta\rho(ax^3 + bx)}{1 - \delta\rho b},\\ y &= \frac{\delta\rho(ax^3 + bx)}{(1 - \delta\rho b)^2} + \frac{\delta\rho x}{1 - \delta\rho b}.\end{aligned} \tag{23}$$

Substituting Eq. (23) into the following equation

$$g(x) = (1+\delta)X - \frac{1}{3}\delta X^3 + \delta Y - x, \tag{24}$$

the following equation can be obtained

$$\begin{aligned}x = (1+\delta)(ax^3 + bx) - \frac{1}{3}\delta(ax^3 + bx)^3\\ + \delta\frac{\delta\rho(ax^3 + bx)}{1 - \delta\rho b} - x.\end{aligned} \tag{25}$$

Let

$$\begin{aligned}g(x) = (1+\delta)(ax^3 + bx) - \frac{1}{3}\delta(ax^3 + bx)^3\\ + \delta\frac{\delta\rho(ax^3 + bx)}{1 - \delta\rho b} - x = 0,\end{aligned} \tag{26}$$

then an eight degree polynomial can be obtained from Eq. (26) as follows:

$$a_9 x^8 + a_8 x^7 + \cdots + a_2 x + a_1 = 0, \tag{27}$$

where $a_1 = (1+\delta)b + \delta(\delta\rho + b)/(1 - \delta\rho b) - 1$.

Obviously, when $a_1 < 0$, then we have the following theorem:

Proposition 6. *If* $a_1 < 0$, *then there must exist a real root* $Z(x,y) \neq O(0,0)$.

It is true that $\|DF^2(Z)\| = \|DF(F(Z))\| \times \|DF(Z)\| \neq 0$ due to $\|DF(F(Z))\| \neq 0$ and $\|DF(Z)\| \neq 0$.

Finally, we obtain the following sufficient conditions for chaos in the sense of Marotto:

Theorem. *If* $0 < \delta < 1, 1/4 < \rho < 1, 0 < b \leq (2\sqrt{\rho} - 1)/\rho$, *together with* $(1+\delta)b + ((\delta^2\rho + \delta b)/(1 - \delta\rho b)) < 1$, *then there exists chaos in the sense of Marotto if one of the following conditions is satisfied*:

(i) $\Delta_0 > 0, \quad |x| < \sqrt{\dfrac{\delta^2\rho b - \delta^2\rho + 2\delta\rho b - 2\delta - 4}{\delta^2\rho b}}$;

(ii) $\Delta_0 = 0, \quad x > \sqrt{\frac{4+\delta-\delta\rho b}{\delta^2}} \quad or$

$$x < -\sqrt{\frac{4+\delta-\delta\rho b}{\delta^2}}; \quad or \quad |x| < \sqrt{1-\frac{\rho b}{\delta}};$$

(iii) $\Delta_0 < 0, \quad x > \sqrt{\frac{\delta^2\rho b - \delta^2\rho + \delta\rho b - \delta}{\delta\rho b - 1}} \quad or$

$$x < -\sqrt{\frac{\delta^2\rho b - \delta^2\rho + \delta\rho b - \delta}{\delta\rho b - 1}}.$$

5. Conclusion

In this paper, the continuous BVP oscillator is transformed into a map-based neuron model by using the forward Euler scheme. As a sensitive bifurcation parameter, the step size plays a fundamental role on bifurcations and chaos of the map-based BVP model. In general, the curve of fixed points, the saddle-node bifurcations, is not affected from the step size. While, the period-doubling bifurcation and the subsequent chaos resulting from it will be much affected with the increase in step size. The larger the step size is taken, then the more irregular is the dynamical behavior of the map-based neuron model and chaos may also occur.

The investigation demonstrates that much attention should be paid on the step size when a map-based neuron model is used as an alternative to a continuous neuron model.

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