

Finding safety in partially controllable chaotic systems

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ABSTRACT

Many discrete-time dynamical systems have a region Q from which all or almost all trajectories leave, or at least they leave in the presence of perturbations that we call *disturbances*. We *partially* control systems so that despite disturbances the trajectories of a dynamical system stay in the region Q at least for some initial points in Q . The disturbances can be thought of as either noise or as purposeful, hostile efforts of an enemy to drive the trajectory out of the region. Our goal is to keep trajectories inside Q despite the disturbances and our *partial control* of chaos method succeeds.

Surprisingly this goal can be achieved with a control whose maximum allowable size is smaller than the maximum allowed disturbance. A fundamental step towards this goal is to compute a set called the *safe set* that had, until now, been found only in certain very special situations.

This paper provides a general algorithm for computing safe sets. The algorithm is able to compute the safe sets for a specified region in phase space, the maximum disturbance value, and the maximum allowed control. We call it the *Sculpting Algorithm*. Its operation is analogous to removing material while sculpting a statue. The algorithm sculpts the safe sets. Our Sculpting Algorithm is independent of the dimension and is fast for one- and two-dimensional dynamical systems. As examples, we apply the algorithm to two paradigmatic nonlinear dynamical systems, namely, the Hénon map and the Duffing oscillator.

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1. Introduction

When trajectories escape. Transient chaos [1,2] is present in many different situations in nonlinear dynamical systems. In such systems, for f a continuous map of phase space to itself,

$$q_{n+1} = f(q_n), \quad (1)$$

there are trajectories that behave chaotically in some region Q of phase space for a while before eventually leaving that region or reaching a stable periodic state. The topological structure inside Q that causes this behavior to occur is a zero-measure set known as a chaotic saddle [2,3].

In various real-life applications, it may be necessary to keep the orbits away from certain regions, that is, to keep trajectories from leaving the region Q . Examples can be found in Refs. [4–6]. Sometimes there are disturbances ξ_n that cause the trajectory q_n to leave Q ; that is,

$$q_{n+1} = f(q_n) + \xi_n. \quad (2)$$

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We always assume $|\xi_n| \leq \xi_0$ for some specified number $\xi_0 > 0$. Our goal is to choose a u_n such that *partially controlled trajectories*

$$q_{n+1} = f(q_n) + \xi_n + u_n \tag{3}$$

can guarantee that q_n remain in Q for appropriate choices of u_n , where the feedback control u_n can be chosen with knowledge of $f(q_n) + \xi_n$. Hence the goal is to find a feedback control algorithm $u(x)$ so that $u_n = u(f(q_n) + \xi_n)$.

We make four assumptions throughout this paper:

1. Q is a closed, bounded region in the phase space.
2. There is a bound $\xi_0 > 0$ such that the disturbances ξ_n in phase space satisfy $|\xi_n| \leq \xi_0$. We say that such ξ_n are **admissible disturbances**.
3. There is a bound $u_0 > 0$ such that the (feedback) control u_n in phase space satisfies $|u_n| \leq u_0$. We say that such u_n are **admissible controls**.
4. The bound of the control to keep the trajectories inside Q is smaller than the bound of the disturbances, that is $u_0 < \xi_0$.

A basic and fundamental ingredient is the use of certain sets referred to as safe sets [7] that are subsets of Q . We will say a set $S \subset Q$ is **safe**, if for each $q \in S$, the distance of $f(q) + \xi$ from S is at most u_0 . That implies there exists an admissible u such that $f(q) + \xi + u$ is in S . We emphasize that whether a set is safe depends heavily on u_0 and ξ_0 . By repeating, we find it is possible to keep the entire trajectory q_n of Eq. (3) in S and hence in Q . Then if q is in a safe set $S \subset Q$, the trajectories can be made to stay in S and therefore in Q by choosing u_n so that q_{n+1} is in S .

An example of a safe set is shown in Fig. 1, for the Duffing oscillator $\ddot{x} + 0.15\dot{x} - x + x^3 = 0.245\sin(t)$ with the smallest possible ratio.

We write ρ for the ratio u_0/ξ_0 and call it the **safe ratio**. If there is a safe set for some choice of ξ_0 and u_0 , then we try to decrease u_0 until there is no safe set and we report that minimum value here where there is a safe set with two-digit precision.

A surprising achievement of the partial control technique is that it allows us to keep trajectories inside a region Q even when the maximum amplitude of the control u_0 is smaller than the maximum amplitude ξ_0 of the disturbances (the noise or attacks of an enemy), even when almost every trajectory of the deterministic system (Eq. 2) leave Q .

Until now, safe sets have only been found for certain one-dimensional maps [8] and for horseshoe maps [7]. Even if horseshoe maps typically arise in nonlinear dynamical systems, finding them is not always a simple task [9].

In practice, we use a grid of points for the region that needs to be partially controlled, and find the largest safe set that will be a subset of that of grid points.

What safe sets look like. We have worked special cases of the problem of finding safe sets for 6 years [7,8,10], and we can now report that we have found an algorithm that always works whenever there is a safe set (always for specified bounds ξ_0 and u_0). It finds the maximum safe set, that is, the largest safe set in Q . We find that maximum safe sets are geometrically more complicated than we expected.

In the second and third sections, we infer the main property that the points belonging to a safe set should satisfy. This will be used to develop the algorithm to compute safe sets. This algorithm is a recursive algorithm that seems to find a safe set whenever there is a chaotic saddle in Q , the region of the phase space considered. To demonstrate our algorithm we have also chosen the Hénon map with a choice of parameters where no periodic attractor exists and all the trajectories diverge after some iterations. Using the algorithm described in the paper, we have found a safe set that can avoid the escape to infinity with a control that is smaller than noise. As an example of a return map of a flow, we have chosen the forced Duffing

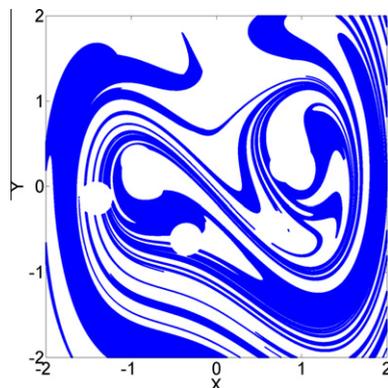


Fig. 1. This is the safe set found for the Duffing oscillator $\ddot{x} + 0.15\dot{x} - x + x^3 = 0.245\sin(t)$ with the smallest possible ratio.

oscillator with a choice of parameters where the Wada property [11] arises in the phase space for all the basins of attraction. We show a safe set that allows the system to behave chaotically for an indefinite time, avoiding the periodic attractors.

2. The “Sculpting Algorithm” for computing the largest safe set in Q .

Given a closed bounded set C and values u_0 and ξ_0 , we declare a point p in C “bad” (for C) if there exists an admissible ξ such that the distance of $f(p) + \xi$ from C is more than the distance u_0 from C . Notice that such $f(p) + \xi$ has no admissible control u for which $f(p) + \xi + u$ is in C . We define the sculpting operator Ψ that cuts away the bad part of C ; that is, $\Psi(C)$ is the set of points in C that are not bad for C .

We iterate the procedure. To find the largest safe set in Q , write $Q_0 = Q$, and $Q_1 = \Psi(Q_0)$, discarding bad points in Q_0 , and $Q_2 = \Psi(Q_1)$, discarding bad point in Q_1 , etc., defining $Q_{n+1} = \Psi(Q_n)$, discarding bad points in Q_n for each $n > 0$. These sets are all compact since f is continuous. It is possible that Q_n is the empty set for some n , in which case there is no safe set in Q . If all Q_n are non empty, then the intersection, defined as $Q_\infty = \bigcap Q_n$ is non-empty since the intersection of nested non-empty compact sets is non-empty.

We claim Q_∞ is a safe set. Let p be in Q_∞ . All we need to do is show that p is not bad for Q_∞ . Since p will not be bad for any Q_n , for each ξ , there will be some point p_n in Q_n such that the $dist(f(p) + \xi, p_n) \leq u_0$. Since the sequence p_n will have a limit point in Q_∞ (since all the sets are compact), $dist(f(p) + \xi, Q_\infty) \leq u_0$. Since that is true for each ξ , p is in Q_∞ .

We claim Q_∞ is the largest safe set in Q . First, notice that any safe set S in Q is a subset of each Q_n and so of Q_∞ , since if p is in S , then for each n , p is not bad for any larger set such as Q_n , so it is in Q_{n+1} , so it is in Q_∞ .

For any set C and distance $d \geq 0$, we write $C + d$ for the set of points that are no more than distance d from C . When C is a set of grid points, we restrict this set $C + d$ to grid points. For any set C and distance $d \geq 0$, we write $C - d$ for the set of points that are in C and are at least distance d from the exterior of C . Then a set S is safe if the set $f(S)$ is a subset of $(S + u_0) - \xi_0$. (In practice, we use a grid and restrict attention to points on the grid.)

Using this notation, we can now write $\Psi(Q_{n+1})$ to be the set of points $x \in Q_n$ for which $f(x)$ is in $(Q_n + u_0) - \xi_0$. Hence the problem essentially reduces to computing the sets $(Q_n + u_0) - \xi_0$, which is not hard in dimension one or two.

3. Implementation of the algorithm

In practice, we use a grid of points on Q , of some thousands of points by thousands points, if phase space is two-dimensional, in a region that includes Q . In all cases the grid size (the distance between nearest neighbor grid points) should be small compared to u_0 . We will at times check grid points beyond this array but the safe set we seek will be restricted to be in this array. We choose Q_0 to be the set of grid points in Q . At each successive step, the subsets $Q_{n+1} = \Psi(Q_n)$ are subsets of the grid. Since the grid is a finite set, for some n we will have $Q_\infty = Q_n$. Indeed, we know we have found Q_∞ when $Q_{n+1} = Q_n$ since Q_m remains the same for all $m > n$.

We write $[[v]]$ for (one of) the closest grid point to $v \in C$. Let V be the set of grid points which are within admissible values of ξ for a given grid point $[[v]]$. We will only work with admissible ξ in V and will call these V -admissible.

For a set of grid points C , we compute $\Psi(C)$ as follows. First, we create the set $C + u_0$ of grid points that are within the distance u_0 of C , as we show in Fig. 2(a). (Here may need to include grid points that are beyond Q .) Second we create the set $(C + u_0) - \xi_0$, that are the grid points in $C + u_0$ with at least a distance ξ_0 from the exterior of $C + u_0$, as shown in Fig. 2(b). We check each p in C as follows. We compute $[[f(p)]]$. Next we determine if $[[f(p)]]$ is in $(C + u_0) - \xi_0$. If $[[f(p)]]$ is not in $(C + u_0) - \xi_0$, is bad, because $[[f(p)]] + \xi$ is too far from C to be pushed into C by an admissible u . This is equivalent to determine if there is a V -admissible ξ such that $[[f(p)]] + \xi$ is not in $C + u_0$. In Fig. 3, we can see schematically which

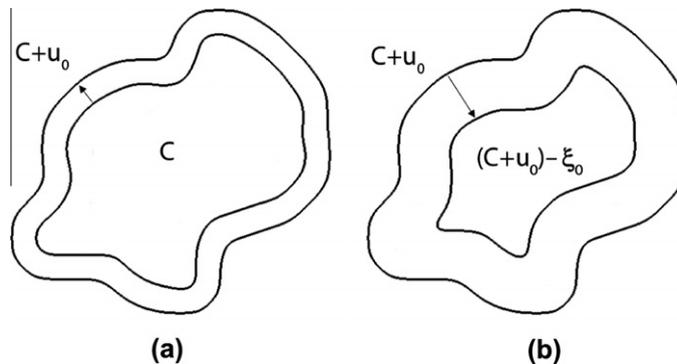


Fig. 2. (a) The set of points $C + u_0$, is the extension of the set C in the grid of points to include also the points that are within a distance u_0 of C . (b) A set C is safe if the set $f(C)$ is a subset of $(C + u_0) - \xi_0$. In practice, we use a grid and restrict attention to points on the grid.

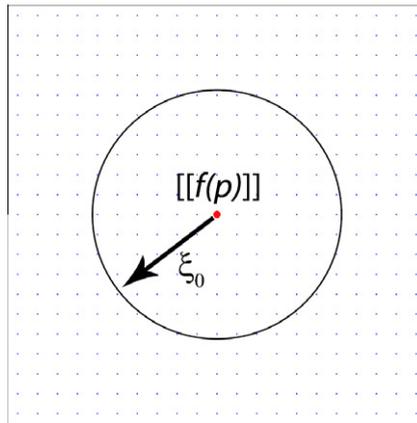


Fig. 3. Here we can see the points of the grid that are within an admissible ξ of $[[f(p)]]$. Here the small points represent the grid points. The big point is $[[f(p)]]$ which is the closest grid point to $f(p)$. And the V-admissibles ξ , are those such that $[[f(p)]] + \xi$ are also a grid point and within the circle of radius ξ_0 .

are the V-admissible ξ for $[[f(p)]]$. If there is one, then p is bad because $[[f(p)]] + \xi$ is too far from C to be pushed into C by an admissible u . Then $\Psi(C)$ is the set of points p in C that are not bad, that is $[[f(p)]] \in (C + u_0) - \xi_0$.

The errors resulting from the use of $[[f(p)]]$ instead of $f(p)$. Let ϵ be maximum distance of any point in Q from the nearest grid point. For example, in dimension 1, that would be half the distance between consecutive grid points. Then $[[f(p)]] - f(p) \leq \epsilon$. In the above algorithm, it is easy to adjust for such errors, so that any point p that might possibly be bad – if we had corrected for such errors – is declared bad. Then $\Psi_1(C)$ can be defined to be the set of points p in C that are not bad and are not possibly bad. The set $\Psi_1(C)$ is a smaller set as a result, more conservative, and the resulting Q_∞ is reliably safe but may be smaller than the true largest safe set in Q . And we can define $\Psi_2(C)$ to be the set of points p in C that are definitely not bad even allowing for errors in substituting $[[f(p)]]$ for $f(p)$. Then $\Psi_1(C) \subset \Psi(C) \subset \Psi_2(C)$. As we sculpt using these, we get upper and lower bounds for Q_∞ . That is, we can first compute Q_∞ using $\Psi_1(C)$ and then $\Psi_2(C)$ separately – using a coarse grid. Then, when we have an idea about Q_∞ from the two estimates we can refine the grid and examine points that are $\Psi_1(C)$ but not $\Psi_2(C)$. We can iterate the process, looking at every finer grids affecting less and less space.

In our computational experiments, the typical smallest safe ratio $\rho = u_0/\xi_0$ that we have found is usually between 0.5 and 0.6 depending on the system. A major factor in computation speed is the number of points of the initial grid in Q_0 . A finer grid means much computation. An increase in the resolution of the initial set increases the probability of finding a safe set. However, if the number of initial points chosen is too big, the algorithm is slowed down considerably. We believe the true largest safe set in Q will be closely approximated by the safe set we find using a grid. Furthermore, we note that our procedure would be valid for maps of any dimension, although the computational effort for the application of the algorithm increases exponentially with an increase in the number of dimensions. If such a set exists, some originality might be required in displaying such a higher dimensional set.

4. Some examples of safe sets

We have tested this new algorithm with two paradigmatic nonlinear dynamical systems, namely, the Hénon map which is a discrete-time dynamical system, and the Duffing oscillator which is a continuous-time dynamical system. The choice of parameters for both are such that they possess a chaotic saddle, which implies the existence of transient chaos.

Hénon map. We consider the Hénon map with a choice of parameters close to the boundary crisis, which occurs for $x_{n+1} = 2.12 - 0.3y_n - x_n^2$; $y_{n+1} = x_n$,

$$\begin{aligned} x_{n+1} &= 2.16 - 0.3y_n - x_n^2 \\ y_{n+1} &= x_n. \end{aligned} \tag{4}$$

For this choice of parameters, we observe that almost all of the initial conditions escape from the square $Q = [-5, 5] \times [-5, 5]$ after a finite number of iterations. The presence of a disturbance in the system typically complicates the survival probability of the orbits inside the square, since a small disturbance can drive the orbit outside the square. If this happens, the orbit would go into the infinity very fast.

To apply the algorithm to the Hénon map, we have chosen this Q as the region of the phase space from which we want to avoid the escapes. This square completely covers the chaotic saddle formed in the parametric region which is close to the boundary crisis.

No sink points exist inside the square, only the saddle points of the chaotic saddle can be found in this region. Then, using the Sculpting Algorithm recursively on the initial set of Fig. 4, we obtain the safe set shown in blue in Fig. 5, after 11

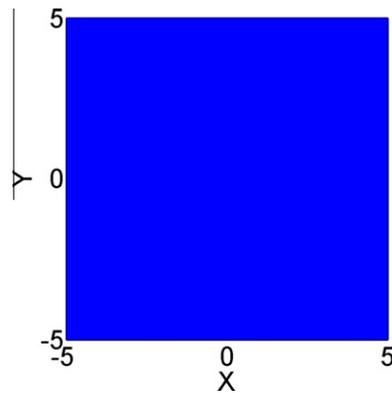


Fig. 4. We use a grid of 3000×3000 points in the square $[-5, 5] \times [-5, 5]$ as our initial set for the Hénon map, $x_{n+1} = 2.16 - 0.3y_n - x_n^2$; $y_{n+1} = x_n$. Applying the Sculpting Algorithm over several iterations we will obtain the desired safe set. We consider that $\zeta_0 = 0.3$ is the worst possible disturbance that we can face.

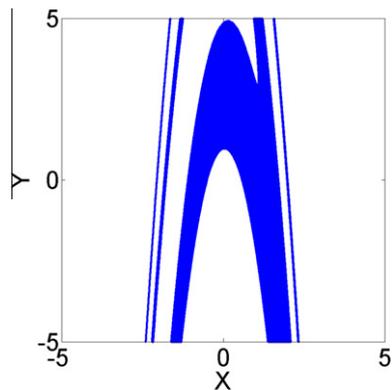


Fig. 5. In this figure we can see the result of applying the Sculpting Algorithm to the Hénon map, $x_{n+1} = 2.16 - 0.3y_n - x_n^2$; $y_{n+1} = x_n$. The safe set appears in blue. The minimum control allowed so that it exits a safe set is $u_0 = 0.18$ for the given disturbance of $\zeta_0 = 0.3$. This is equal to a ratio of $\rho = 0.6$.

iterations of Ψ . Moreover, the safe sets are mapped in such a way that the images are surrounded by the safe set itself, as is expected.

In the simulation that we have made with the Hénon map to obtain Fig. 5, we have used a value of $\zeta_0 = 0.3$ for the bounded disturbance. For this value, the minimum control bound (to two-digit precision) for which there is a safe set is $u_0 = 0.18$. Of course, if for the same value of the disturbance the control allowed were higher, the safe set found would be a little larger. The minimum safe ratio obtained for this particular case is $\rho = u_0/\zeta_0 = 0.6$.

Duffing oscillator. Now we demonstrate the algorithm with the Duffing oscillator with this choice of parameters:

$$\ddot{x} + 0.15\dot{x} - x + x^3 = 0.245\sin(t). \quad (5)$$

With these parameters, a very interesting topological property appears here. This is the Wada property. Due to this property, every point on the boundary of any basin is also on the boundary of the other two basins. [11]. This is what we see in the Fig. 6. With this configuration, the Duffing oscillator has a region that shows a transient chaotic behavior in the square $[-2, 2] \times [-2, 2]$ due to the presence of a chaotic saddle. For this choice of parameters, the system possesses two period-1 orbits and one period-3 orbit. We can see this in the Fig. 7(a).

The idea of applying the partial control technique to the Duffing oscillator is slightly different than that of using it in the Hénon system. The region Q in this case contains several attracting periodic orbits that will eventually attract almost every trajectory. Our goal here is to have the trajectories partially controlled so that they stay away from the attracting fixed points and the attracting periodic orbit of period 3. The unperturbed, uncontrolled behavior of the system exhibits transient chaotic behavior. The orbits behave chaotically, but after some time, the orbits fall close enough to some of the stable periodic attractors, showing an intermittent behavior in the presence of the disturbances.

The upper bound of the disturbance that we consider in this system is $\zeta_0 = 0.08$. The situation changes drastically if we use the partial control technique. Then it is possible to maintain the chaotic behavior indefinitely, with a control smaller than the disturbance, avoiding the intermittency. We have found that it is possible to achieve this with a ratio of control versus noise of approximately 0.59. For u_0 significantly smaller than 0.0475, there is no safe set.

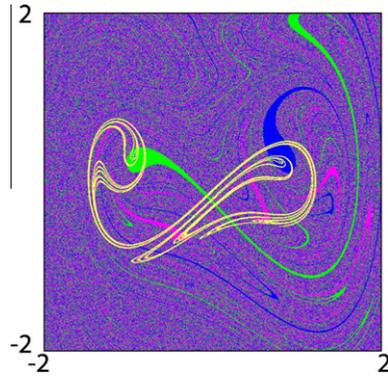


Fig. 6. In this figure we show the complex structure of the phase space for the Duffing oscillator $\ddot{x} + 0.15\dot{x} - x + x^3 = 0.245\sin(t)$. In this system are present three different basins of attraction (magenta, blue and green) which have the Wada property. The invariant unstable manifold associated to the chaotic saddle appears in yellow. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

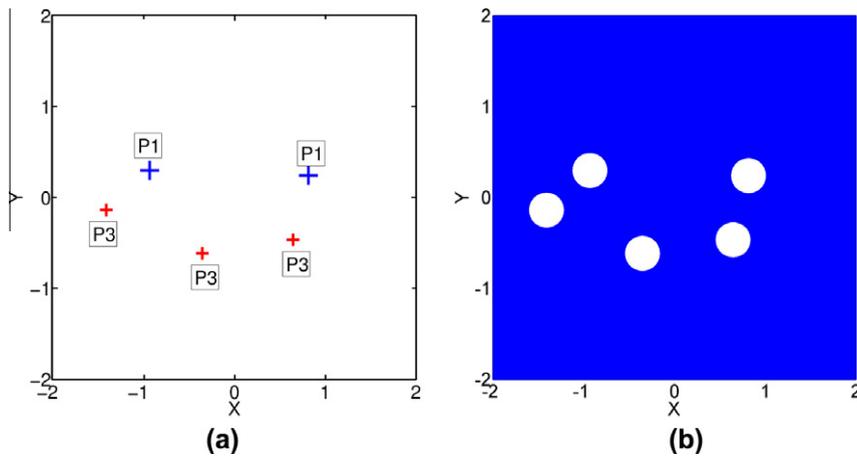


Fig. 7. (a) In this figure are represented the periodic attractors that has the Duffing oscillator $\ddot{x} + 0.15\dot{x} - x + x^3 = 0.245\sin(t)$: two period-1 attractors and one period-3 attractor. We also shown with circles of radius 0.2 the region of the phase space that we want to avoid whatever the disturbances. (b) We use a grid of 6000×6000 points in the square $[-2, 2] \times [-2, 2]$ as our initial set, but removing the zones that we want to avoid, that is the circles. Applying the Sculpting Algorithm over several iterations we will obtain the desired safe set. We let $\xi_0 = 0.08$ be the maximum size of the vector perturbation.

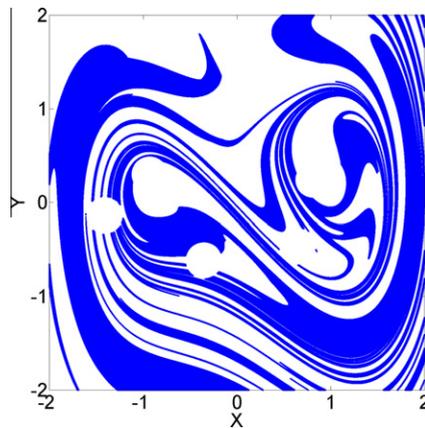


Fig. 8. In this figure we can see the result of applying the Sculpting Algorithm to the Duffing oscillator $\ddot{x} + 0.15\dot{x} - x + x^3 = 0.245\sin(t)$. The safe set appears in blue. The minimum control allowed, so that it exits a safe set is $u_0 = 0.0475$, with a maximum disturbance of $\xi_0 = 0.08$. This is equal to a safe ratio $\rho \approx 0.59$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

To apply the partial control technique to the Duffing oscillator, we need to clarify which is the concept of escape in this case. The region from which all the trajectories escape will be the square $[-2, 2] \times [-2, 2]$ minus certain holes around the periodic attractors. We say there is an escape here if a given trajectory enters one of the circles or if it leaves the square. Then we use a grid of 6000×6000 points in the square $[-2, 2] \times [-2, 2]$ as our initial set as in Fig. 7(b), but removing the zones that we want to avoid, that is the circles.

Finally, applying the Sculpting Algorithm to the set plotted in Fig. 7(b), we obtain the safe set of the Fig. 8 in 15 iterations of Ψ , where the safe set appears in blue.

5. Conclusions and discussion

We present a general algorithm for finding safe sets (whenever one exists) for any continuous bounded discrete-time dynamical system, in order to apply the partial control technique. Such safe sets can be found for example inside those regions from which trajectories escape after having some complicated dynamical behavior. We call it Sculpting Algorithm, as an analogy to removing material while sculpting a statue. At this point, there is no general mathematical result that guarantees the existence of a safe set, given a dynamical system, aside from running the Sculpting Algorithm on an example. Thus, our algorithm opens the door for a wider application of partial control to discrete-time dynamical systems.

Our numerical simulations suggest that the safe sets are close to the invariant stable manifold of the chaotic saddle, wherever it is hyperbolic, that is, where the dynamics is more “similar” to that of a horseshoe map. We can guarantee that if a safe set exists for a given situation, it can be found using our algorithm.

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