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Regularization of Map-based Neuron Models Using Phase Control

Javier Used, Alexandre Wagemakers, Miguel A.F. Sanjuán †

Nonlinear Dynamics, Chaos and Complex Systems Group, Departamento de Física, Universidad Rey Juan Carlos, Tulipán s/n, 28993 Móstoles, Madrid, Spain (E-mail: javier.used@urjc.es, alexandre.wagemakers@urjc.es, miguel.sanjuan@urjc.es)

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Abstract

Recently discrete dynamical systems, maps, have been also used as valid phenomenological neuron models, and are able to furnish an advantageous alternative to continuous dynamical systems for the modelling of the spiking behavior of single neurons and of neuronal networks as well. Periodic and chaotic spiking, phasic and tonic bursting, subthreshold oscillations and many more specific features of the activity of real neurons can be reproduced by maps with a minimum of analytical complexity. As an external stimulation is applied to the neuron, its response can be of two different natures: periodic or erratic. We present a simple method of control that allows to choose one of the possible responses when the perturbation is periodic. The phase difference between the periodical driving and the control plays a decisive role.

1 Introduction

A lot of work has been done in the past few years on modelling neuronal dynamics using methods derived from nonlinear dynamics. Most of the well-known models to describe neuronal dynamics are based on ordinary differential equations [1].

However, there are also in the literature many other neuronal models which are described with the help of discrete dynamical systems [2]. Computational, analytical and theoretical factors can indeed make discretization advisable. Even though this can be done in different ways, discretization in time transforms the ODEs into discrete dynamical systems, or maps. Discrete-time models have a long tradition in the physics of complex systems [3], and in the field of artificial neural networks, but it is only recently that they have begun to receive more attention as valid models of biological neurons. Among the different map-based

†Corresponding author.
Email address: miguel.sanjuan@urjc.es

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neuron models, the chaotic Rulkov neuron model [2, 4] is a simple two-dimensional nonlinear map which possesses many interesting features of the real biological neurons. Despite its mathematical simplicity, many effects observed in neuronal cells are qualitatively contained in it. For a given parameter set, the neuron is in a quiescent state when not subjected to an external perturbation. When an external forcing of sufficient intensity is applied, the map can switch to an oscillatory regime and returns to its quiescent state as soon as the perturbation recedes. The response of the system is different, depending on the values of the parameters that appear in the model. The two typical oscillatory regimes that mimic neuronal behaviors are tonic spiking, where a series of sustained pulses appears reminding the spike train of a neuron, and autonomous bursting, where a brief train of short pulses alternates with a silent phase. In both regimes, we can observe a chaotic behavior of the time series which is generated by the nonlinear function of the model. The forcing can be a constant external input or a periodic stimulation such as a periodic neuronal input [15], or even a combination of both. If we look closer at the response when the input is a periodic forcing, we observe that the series of bursts (or spikes) can be either periodic or erratic. That is, we can say as a general statement that the response over a long time interval is either periodic or chaotic.

It may be of interest to select the particular response when the system receives such an external signal, that is, there might be necessary to control its behavior [16]. Several techniques to control nonlinear dynamical systems have been proposed, including both feedback [5] and nonfeedback methods [6]. Feedback methods, although more effective, require a fast and accurate response to work appropriately, while nonfeedback methods only depend on an adequate choice of the parameters.

In our case, the system can be controlled if a second periodic forcing with the same frequency is added up. In this scheme the control parameter is the phase difference between the main and the second periodic forcing applied to the dynamical system. This phase difference is critical for the overall behavior. The technique is named as the phase control [6, 11]. Such example of control has been applied successfully to a wide range of different dynamical systems, such as nonlinear oscillators [7–9], a CO_2 laser [10], or even simple continuous neuron models [11].

In spite of its simplicity, the technique has been scarcely applied to discrete dynamical systems [17]. This is precisely what we do in this paper: to apply this technique to the chaotic Rulkov model. Since we set the neuron in a particular regime, the application of the phase control technique can be useful to drive the system into a different dynamical regime.

2 The Chaotic Rulkov Neuron Model

The Rulkov neuron model is a good alternative to the continuous time neuron models. It has a simple and elegant formulation while keeping interesting dynamical regimes such as spiking and bursting [2, 4]. The discrete time dynamics simplifies the analysis and the simulations of the model since the numerical integration is straightforward and very fast on modern computers. These features make this model an ideal candidate for the numerical simulations when the physiological details are not crucial.

The Rulkov map is an abstract mathematical model, although it shares some specific features with others neuron models closer to experimental observations. The two variables reflect the two important time scales of a neuron model. The variable x represents the fast dynamics of the system that usually models the membrane voltage of the neuron, whereas y is the slow variable and represents the variations of the ionic recovery currents. Finally, the term I_n embraces the sum of the external influences on the neuron.

The two-dimensional map proposed by Rulkov [4] is:

$$x_{n+1} = \frac{\alpha}{1+x_n^2} + y_n + I_n, \tag{1}$$

$$y_{n+1} = y_n - \eta(x_n - \sigma). \tag{2}$$

These equations present a variety of behaviors depending on the control parameters α and σ. The two typical regimes are the spiking regime: a series of sustained pulses appears reminding the spike train of a neuron; and the bursting regime: a brief train of short pulses alternates with a silent phase. An interesting point is the chaoticity of the orbit in both regimes, which is due in part to the shape of the nonlinear function in the equation corresponding to the variable x. When the variable x is oscillating without bursts or spikes it is said to be in a subthreshold regime. We describe shortly the role of each parameter of the model:

- The regime can be changed with the parameter α, which is critical for the type of dynamics of the neuron. The bursting oscillations appear for a small range of parameters $4 < \alpha < 4.5$ [4], above this value chaotic spiking can occur. The bursting dynamics consists of an oscillation of the variable x between a stable equilibrium state and a fast chaotic orbit. This dynamics appears frequently among neurons and is well known for neuroscientists and modellers [12]. A representative orbit of the variable x of the chaotic neuron model is shown in figure 1.

- The parameter η represents the time constant of the variable y, which evolves very slowly compared to the variable x. It corresponds to the slow recovery variable that drives the system back to a stable equilibrium point, while the fast variable oscillates. This parameter is usually kept very small in order to obtain slow variations. We will use $\eta = 10^{-4}$ throughout the paper.

- The parameter σ is a threshold value that is important for the variations of the variable y. While the value of x remains below (above) the value of σ, the variable y increases (decreases) slowly. If we set the value of σ low enough, the system rests on a stable equilibrium point in absence of an external current I_n. In this case however, if a short external perturbation is added to the signal, the neuron displays a single burst or spike. The neuron is said to be in an excitable regime. It is a particularly interesting regime since biological neurons are most of the time in an excitable state.

We have chosen here the parameters σ, η and α in order to obtain an excitable bursting regime.

The variable I_n in the model represents the external influences as for example a periodic forcing $I_n = B\cos(2\pi\omega n)$. This is a simple way to model a periodic stimulus that drives the neuron. The response of the system can be of a different nature depending on the frequency and amplitude of this forcing. We focus here on the specific response of the model depending on these two parameters and the conditions that allows controlling its behavior.

3 Bursting Regularization

While staying in the subthreshold regime, the Rulkov map is not showing any bursting activity. However if a periodic current stimulates the neuron strongly enough, a train of bursts activity takes place. The behavior of the map differs depending on the parameters of the stimulus, mainly its frequency and amplitude. In order to quantify this response, we study the periodically forced Rulkov map:

$$x_{n+1} = \frac{\alpha}{1+x_n^2} + y_n + B\cos(2\pi\omega n) \tag{3}$$

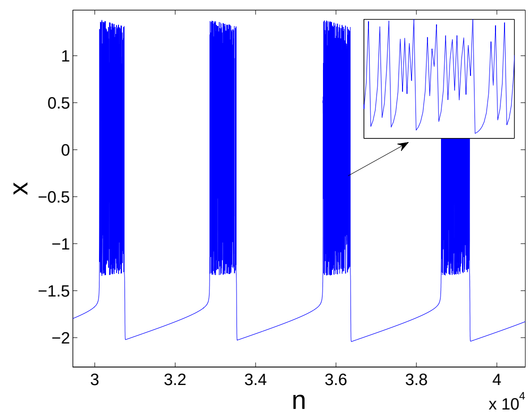


Fig. 1 Example of a typical chaotic bursting of the model described by equations $x_{n+1} = \alpha/(1+x_n^2)+y_n+I_n$; $y_{n+1} = y_n - \eta(x_n - \sigma)$. In the main figure we observe the alternance between the quiet states and the fast chaotic oscilations. In the inset, we show a detail of one of the bursts where the chaotic nature is clear.

$$y_{n+1} = y_n - \eta(x_n - \sigma). \tag{4}$$

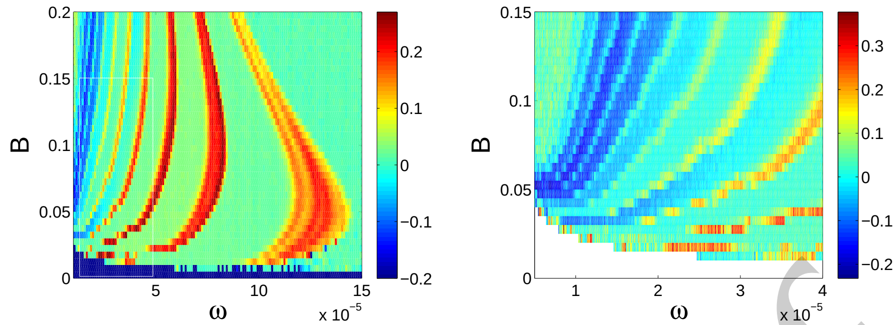
The periodic forcing consists of a sinusoidal wave of amplitude B and angular frequency ω. As we have seen in the previous section, the dynamics for a given set of parameters can correspond to a subthreshold regime (also known as the silent regime), or the bursting regime. If the second case occurs, we observe two different behaviors for the time occurrence between two bursts: periodic or erratic. In some situations, the external input cannot be chosen or removed, and it therefore determines the behavior of the system. We will show that this is not entirely true as we might select the regime by adding up another external signal.

Now, we focus our attention on the time elapsed between one burst and the following one with a measure called inter burst interval (IBI). This is a similar technique as the one described by Sauer et al. [13] for measuring the time intervals between spikes in a time series. The method employed to detect these time intervals consists of tracking the period of the bursts:

- First, the time series is filtered with a low pass filter in order to remove or atenuate the fast variations of the spiking activity.
- The obtained time series is then transformed into a square-waveform with a simple threshold.
- The times of the rising-edge front of the square-waveform are stored into a buffer. These are the time occurrences of the bursts.
- With the previous time vector, we compute the time difference between occurrences. This is our IBI time series.

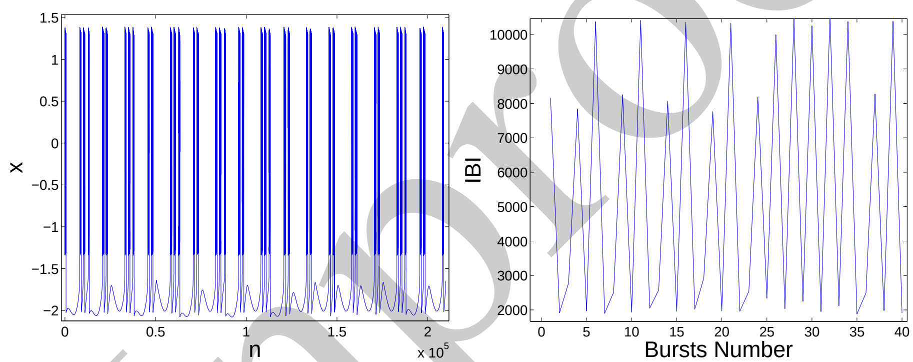
The IBI time series is then processed with the numerical computation package "TISEAN" [14] in order to estimate the maximal Lyapunov exponent of our simulations.

In order to specify the behavior of the system with a periodic forcing, we compute the exponent of the IBI varying the frequency and the amplitude of the external forcing for the following parameters: $\sigma = -1.65$, $\alpha = 4.15$, $\eta = 10^{-4}$. The results are shown in figure 2 for a span of B and ω values. The regions with a negative Lyapunov exponent, meaning a periodic or subthreshold behavior, are located for small amplitudes



(a) (b)

Fig. 2 The figure shows the maximal Lyapunov exponent of the IBI time series in function of the forcing frequency ω and the forcing amplitude B. We can see large regions of periodic or stable behavior (dark colors), and regions with chaotic behaviors (light colors).The bottom figure corresponds to the region marked by a white rectangle inside the upper figure.



(a) (b)

Fig. 3 (a) Time series of the bursting neurons while the parameters of the forcing are in a chaotic regime ($\omega = 8 \cdot 10^{-5}$ and $B = 0.1$). In (b) we show the corresponding IBI time series.

of the forcing term, below $B = 0.02$, and for small frequencies, below $2 \cdot 10^{-5}$. The light color areas mean that the intervals between bursts are irregular or chaotic.

If we look closer at the bursts train, for example with the forcing parameters $\omega = 8 \cdot 10^{-5}$ and $B = 0.1$, the bursts occurrences seem irregular as shown in figure 3 (a). The corresponding IBI time series is shown in figure 3 (b). The Lyapunov exponent of this time series is positive, thus it confirms the intuition about the irregularity of the time series. Such aperiodic behavior appears to be very common when the system receives a periodic forcing as shown in figure 2. However, we will show that it is possible to return to a periodic bursting regime by simply applying the phase control technique to our periodically forced map. For this purpose, we introduce a small perturbation with the same frequency into the system in an additive way:

$$I_n = B\cos(2\pi\omega n) + Bk\cos(2\pi\Omega n + \phi). \tag{5}$$

We choose equal frequencies $\omega = \Omega$ in a first attempt to apply the technique. We consider that the amplitude

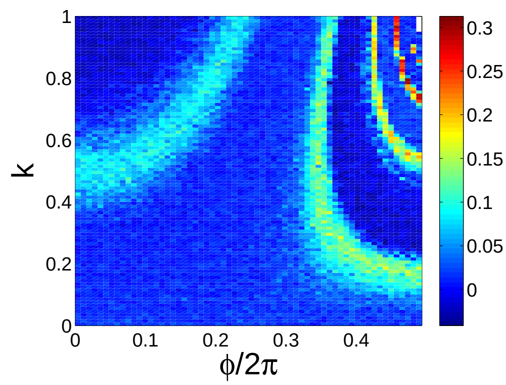


Fig. 4 Phase control of the system while it presents chaotic behavior. The phase and the amplitude of the control are varied in order to obtain a periodic behavior depicted by the dark blue regions. The color scale is proportional to the maximal Lyapunov exponent of the IBI time series. We have chosen $\omega = 2.7 \cdot 10^{-5}$ and $B = 0.09$.

of this second perturbation is Bk, with $k \in [0,1]$ so that it becomes a fraction of B. We do it so, since we are mainly interested in using a control of smaller amplitude that the original forcing. The phase ϕ can be varied over $[0,2\pi]$; however there is a symmetry with respect to π that allows us to save computation time by observing only the interval $[0,\pi]$.

When the phase control is applied to the system, the behavior can change dramatically. In figure 4, the phase and the amplitude of the control affects the system in such a way that for some choices of the phase difference ϕ and amplitude k the system is driven back to a periodic regime, which corresponds to dark colors in the figure 4. It is worth noticing that for values of $k > 0.5$, we can always control our system by choosing an adequate phase difference. The mechanism by which the phase control works on the system can be understood with a simple analysis. The forcing terms of the system can be split in order to make appear a cosine and a sine contribution. Assuming equal frequencies for both forcings we obtain:

$$\begin{aligned} f(n) &= B\cos(2\pi\omega n) + Bk\cos(2\pi\omega n + \phi) \\ &= B(\cos(2\pi\omega n)(1 + k\cos\phi) - k\sin\phi\,\sin(2\pi\omega n)). \end{aligned} \tag{6}$$

The first term, which depends on $\cos(2\pi\omega n)$, dominates for small values of ϕ, meaning that the second oscillating term in $\sin(2\pi\omega n)$ is negligible. If we assume that the forcing term is roughly:

$$f(n) \simeq B(1 + k\cos\phi)\cos(2\pi\omega n), \tag{7}$$

then the effect of the second periodic forcing is mainly to shift the system into a regime where the forcing is of a different amplitude. If this effect is strong enough, then the control drives the system into a periodic regime again.

Until now, we have applied the control varying the phase and the amplitudes of both forcings. However, we can also change the frequency Ω of the second periodic forcing, by choosing for example multiples of ω. In figure 5 we show the result of applying the phase control to the system with the same parameter set when the frequency of the second forcing is twice the frequency of the initial forcing ($\Omega = 2\omega$). The phase difference between the two signals is varied this time over $[0,2\pi]$ since the symmetry is broken due to the different frequencies of both signals. It can be appreciated how the system is also controlled with the new frequency for a wide range of values of the amplitude and the phase. We consider here a new periodic

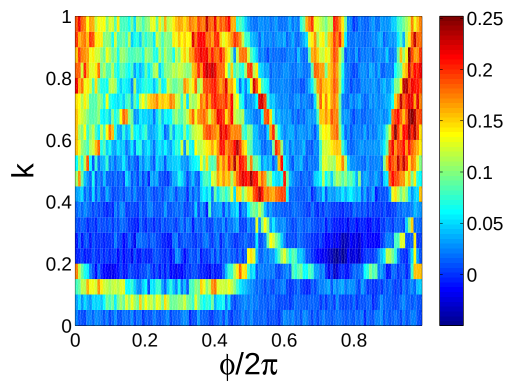


Fig. 5 Phase control of the system with a frequency $\Omega = 2\omega$. The system appears to have periodic bursting when the second periodic forcing is applied. For example, with a second forcing with $k = 0.7$, the system exhibits maximal Lyapunov exponents lower or equal than 0 for an appropriate choice of the phase ϕ. The choice of parameters for this simulation is $\omega = 2.7 \cdot 10^{-5}$ and $B = 0.09$.

forcing with $k = 0.7$, $\omega = 2.7 \cdot 10^{-5}$ and $B = 0.09$. The computation of the maximal Lyapunov exponents associated to the periodically forced map for different values of k and ϕ is shown in figure 5. As in previous figures, we can observe that there exists pairs of (k, ϕ) values for which the control can be achieved, that is, we can find orbits with negative Lyapunov exponents.

4 One Dimensional Model

A major simplification for the Rulkov model consists of using only one of the two equations for the dynamics. As we set the slow variable on the edge of the saddle-node bifurcation, the fast variable behaves as a chaotic excitable neuron. A simplified model is then

$$x_{n+1} = \frac{\alpha}{1+x_n^2} + \gamma + I_{ex}. \tag{8}$$

The parameter γ has to be chosen carefuly in order to get an excitable regime for this model. Keeping $\gamma \leq$ -2.76 is a necessary condition for $\alpha = 4.15$, since for $\gamma > -2.76$ autonomous oscillations can appear due to birth of stable chaotic orbits in the system [18]. In the excitable regime, when an external continuous current I_{ex} is applied to the map, chaotic oscillations appear as a consequence of the saddle-node bifurcation. The dynamics of the map reminds the spiking activity of a neuron as shown in figure 6.

If the external current is a periodic forcing in the form $I_{ex} = B\cos(2\pi\omega n)$, we obtain also chaotic and periodic behaviors depending on the frequency and amplitude of the forcing. In order to quantify this effect, we compute the maximal Lyapunov exponent of the x time series for a span of forcing frequencies ω and amplitudes B in figure 7. Dark colors mean a periodic or stable behavior, while light colors mean a chaotic oscillating activity.

We aim now at controlling this chaotic spiking behavior in order to get periodic oscillations when the system is forced externally. The control is introduced as an additive forcing with a frequency Ω, a phase ϕ and amplitude kB

$$x_{n+1} = \frac{\alpha}{1+x_n^2} + \gamma + B\cos(2\pi\omega n) + kB\cos(2\pi\Omega n + \phi). \tag{9}$$

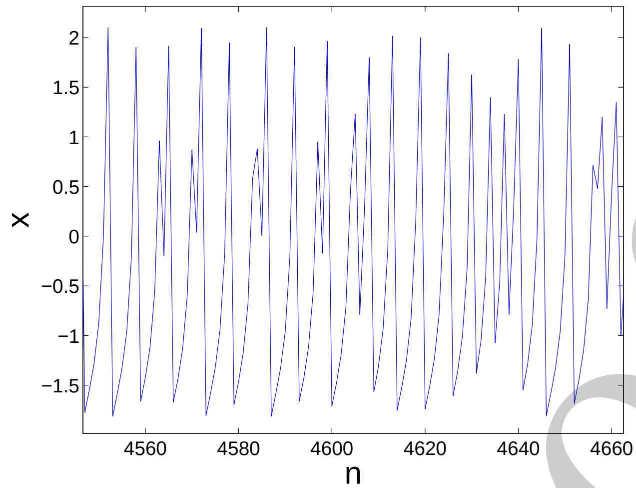


Fig. 6 Dynamics of the one dimensional Rulkov map with parameters $\alpha = 4.15$, $\gamma = -2.85$, $I_{ex} = 0.3$. The orbit is chaotic and is similar to the chaotic dynamics of the two dimensional Rulkov model in the bursting regime.

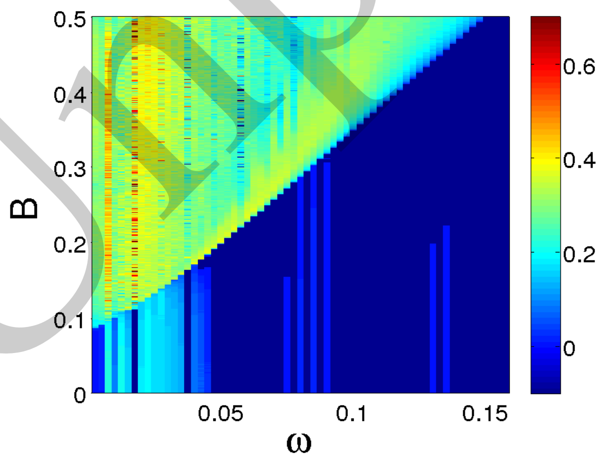


Fig. 7 The figure shows the maximal Lyapunov exponent associated to the map $x_{n+1} = \alpha/(1+x_n^2)+\gamma+I_{ex}$ in terms of the amplitude B and the frequency ω when only a single external forcing is acting on the system. The black regions represent the periodic regime of the dynamical system. This system can be controlled even with very small amplitudes of the control signal.

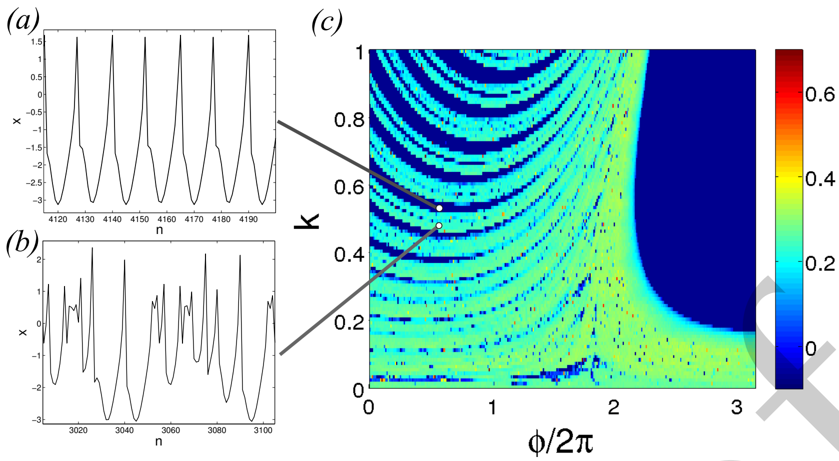


Fig. 8 In figure (c) Maximal Lyapunov exponent of the time series associated to the map $x_{n+1} = \alpha/(1+x_n^2)+\gamma+I_{ex}$ once the phase control is applied, in terms of parameters k and ϕ. The dark regions mean a controlled behavior of the system, either periodic or subthreshold. In (a) we show an example of a time series for $k = 0.58$ and $\phi/2\pi = 0.6$ that is clearly periodic. In the figure (b) we have a time series of the same system for parameter values $k = 0.55$ and $\phi/2\pi = 0.6$, where the orbit is now chaotic.

The parameter k of the amplitude is also kept between $[0,1]$ in an attempt to keep the control very small and we consider the frequency Ω to be equal to the frequency ω of the first forcing. For a given forcing, for example $B = 0.35$ and $\omega = 0.08$, we apply the control for different values of k and ϕ. The results appear in figure 8 (c) where we show the maximal Lyapunov exponent in function of k and ϕ. The phase control technique works well also for this system driving it back to a periodic regime for a large set of (k,ϕ) parameters. The map can be controlled given a driving signal only a fraction smaller than the original forcing, that is, with a forcing of amplitude $kB = 0.1$ we can already control the system. In the figure 8 (a) and (b) we plot the time series of the variable x for two very close parameter sets k and ϕ: $k = 0.58$ and $\phi/2\pi = 0.6$ in Fig (a) and $k = 0.55$ and $\phi/2\pi = 0.6$ in figure (b). The two time series are however very different in comparison, since the first one is periodic and the second one is chaotic. Such a result shows that the phase control method can drive efficiently the system, thus we can choose the dynamical regime of the system by adjusting only two parameters.

5 Conclusions

In this work we have applied the phase control technique to a simple nonlinear map that mimics the behavior of a biological neuron. The interest of this technique lies in its simplicity, where a control can be easily added to the fast variable. In our model, the time interval between bursting activity can be chaotic depending on the frequency and amplitude of the external forcing. As the control is switched on, the intervals between bursts become periodic. This is what we called bursting regularization.

Finally, we demonstrate with an even simpler system, a one-dimensional map, that the phase control can be succesfully applied in order to achieve the regularization of the orbits. If the time series of this map is chaotic for a certain amount of forcing, the phase control can once again drive the system into a periodic regime and vice versa.

What we have shown by applying the phase control technique to this map-based neuron model, can be applied to a wide range of discrete dynamical systems. Since the Rulkov map can be understood as a discrete-time paradigm for the study of the dynamics of neurons, the same ideas can be also applied to other similar nonlinear maps modelling biological phenomena.

6 Acknowledgements

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