Frequency dispersion in the time-delayed Kuramoto model

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We study the synchronization and frequency distribution in networks of time-delayed Kuramoto oscillators with identical natural frequency. It is found that a pronounced frequency dispersion occurs in networks with nonidentical degree distributions. The deviation of the average network frequency from its natural frequency, induced by the time delay, is identified as a necessary component for this phenomenon. Altogether this results in states intermediate between perfect synchronization and complete incoherence.

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I. INTRODUCTION

For many years, the Kuramoto model has served as a kind of prototype for the study of various synchronization phenomena among coupled oscillators. The Kuramoto model has already been studied in many different versions [1], including time-delayed coupling [2–7], which is the topic of the present paper. The set of equations studied in the original work of Kuramoto [8] looked as follows:

$$\dot{\phi}_i(t) = \omega_i + K \sum_{j=1}^{N} \sin[\phi_j(t) - \phi_i(t)]. \qquad (1)$$

This constitutes an all-to-all coupled system where the natural frequencies ω_i of each oscillator are drawn from some symmetric probability distribution $g(\omega)$. The question raised was: At which coupling strength K does the incoherent state becomes unstable to synchronization? It was found that when the coupling strength K passes a certain critical value K_c those oscillators with natural frequencies sufficiently close to the mean $\langle \omega_i \rangle$, which for strong enough coupling includes all oscillators, will lock together at a common frequency while the others will be drifting at their respective frequencies. Altogether, this gives rise to a state intermediate between perfect synchronization and complete incoherence.

Starting from another end, a sparsely connected network of time-delayed Kuramoto oscillators was analyzed in an exact manner by Earl and Strogatz [3]:

$$\dot{\phi}_i(t) = \omega + \frac{K}{k} \sum_{j=1}^{N} a_{ij} \sin[\phi_j(t-\tau) - \phi_i(t)]. \qquad (2)$$

Here τ is the time delay and a_{ij} are coefficients that take the value 0 or 1. The number of nonzero coefficients a_{ij} for a fixed i is called the degree d_i of the oscillator ϕ_i and represents simply the number of incoming connections. We will always assume that the normalizing factor k is equal to the mean degree, that is $k = \langle d_i \rangle$.

In the case of time-delayed interactions you may ask for example: At which time delay does the completely synchronized state loose its stability? If, as in our case, all the oscillators have just one and the same natural frequency ω, then one might expect an abrupt shift from perfect synchronization to complete incoherence as the time delay τ increases. Indeed, this is typically the case if we consider a network of time-delayed Kuramoto oscillators with identical natural frequency and identical degree distribution, (which means that ω and d_i are the same for all oscillators). Moreover, for identical degree distribution it turns out that other aspects of the topology of the network plays no role in determining the critical time delay at which the in-phase synchronized state $\phi_i(t) = \Omega t$ (where Ω is a common fixed frequency) loses its stability [3].

In this study we will compare this latter system with one where ω is still the same for all oscillators but where we allow a nonidentical degree distribution over the network. As we will see, this gives rise to states of intermediate synchronization, which are somewhat reminiscent to those analyzed by Kuramoto. In doing this, we might also come closer to answer another pertinent question in the area of time-delayed interactions, namely: What are the actual mechanisms behind synchronization and decoherence? In a previous study [7], a link between the phase synchronization and the average frequency was found in certain time-delayed neuron networks as well as in networks of time-delayed Kuramoto models. Here, we will expand further on this theme and demonstrate how, in fact, the frequency plays a clearly distinguishable role in the transition between synchronization and incoherence.

Our focus will be on the behavior of the system given by Eq. (2) where the network has a *binomial* degree distribution. The latter was constructed by first choosing the degree of each oscillator d_i from a binomial distribution with mean 4 and standard deviation $2\sqrt{1 - 4/N}$, where N is the number of oscillator in the network. Given this number d_i, the indices j for which $a_{ij} = 1$ (counting to a total of d_i) was chosen randomly. Numerical simulations reveal that for this system we no longer see an abrupt change from perfect synchronization to almost complete incoherence at a certain time delay, but rather a gradual decline in synchronization as a function of time delay. Moreover, this is accompanied by a phenomenon which we chose to call *frequency dispersion* (this nomenclature was also used for example in Ref. [9]). Despite the fact that each oscillator is given the same natural frequency ω, in the stages of intermediate synchronization, the oscillators do not proceed with the same time-averaged frequency. The main purpose of this paper is to explain how these various phenomena are intertwined.

II. SYNCHRONIZATION, AVERAGE FREQUENCY, AND FREQUENCY DISPERSION

The synchronization order parameter introduced by Kuramoto is given by the following expression:

$$r(t) = \frac{1}{N} \left| \sum_{j=1}^{N} e^{i\phi_j(t)} \right|, \qquad (3)$$

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where N is the total number of oscillators. The time t is inserted here explicitly to distinguish the momentary order parameter $r(t)$ from the time-averaged order parameter r to be introduced below. For systems finite in size and where the oscillators propagate at different frequencies you cannot expect the synchronization order parameter $r(t)$ to reach a value constant in time. For this reason, we need to introduce some kind of time-averaged order parameter, which for convenience we simply denote by r. Since $r(t)$ is strictly positive, one might think that the most straightforward way to do this is to simply integrate $r(t)$ over time and then divide by the length of the time interval. The latter definition has been used in several works, including our previous study [7]. However, in this paper we have chosen instead to calculate the root mean square of the order parameter $r(t)$:

$$r \equiv \sqrt{\langle r^2(t)\rangle_T}, \tag{4}$$

$$\langle r^2(t)\rangle_T \equiv \frac{1}{T}\int_0^T r^2(t)dt. \tag{5}$$

There are some ways to argue why this might be a more proper way to do the time averaging under conditions where $r(t)$ never reaches a constant value over time. One reason is that the root mean square is typically much easier to treat analytically and might therefore, in certain cases, facilitate a comparison between analysis and simulations.

Moving on, the average frequency of each oscillator is defined as

$$\Omega_i \equiv \frac{1}{T}\int_0^T \dot{\phi}_i(t)dt, \tag{6}$$

where the integration is over the entire time interval T. If we let $\Omega = \langle \Omega_i \rangle$ be the average frequency of all oscillators, the frequency dispersion is defined accordingly as

$$\sigma^2 \equiv \langle (\Omega_i - \Omega)^2 \rangle, \tag{7}$$

where the right-hand side denotes the average taken over all the values of Ω_i. Notice in particular that we are not concerned here with the momentary dispersion in frequency. With the above definition it is only necessary to know the initial and final values of each phase variable ϕ_i.

The computer simulations have been performed with a modified sixth-order Adams-Bashforth-Moulton integration scheme with an iterated corrector formula [2,10]. This integration method is useful when the number of differential equations in the system is large, which is typically the case in coupled networks of oscillators. The integrator has been compared with other well known methods with a high degree of precision. The step size for the following simulations was set to $dt = 0.01$.

In Fig. 1 we show the results of the computer simulations of Eqs. (2) performed on networks consisting of 300 oscillators, one class of networks with identical degree distribution (from now on denoted as the uniform network) and another class of networks with binomial degree distribution (denoted as the nonuniform network). For maximum comparability, we have chosen the total number of connections (i.e., the number of nonzero a_{ij}) to be the same in all cases. For all networks the average node degree $\langle d_i \rangle$ was put equal to 4. For the uniform network the choice of initial conditions is crucial: If you give the oscillators initial values (including initial history) close to an in-phase synchronized state then the simulations corroborate the analytical results of Earl and Strogatz [3], where complete synchronization is obtained over a much wider range of τ than would be the case for other initial values. Here instead we have chosen initial conditions close to a completely incoherent state (for example $\phi_j(0) = 2\pi j/N$ with the initial history computed accordingly). As can be seen in Fig. 1, for the uniform network (dashed line) we see a rather abrupt change from complete synchronization to incoherence accompanied by an abrupt change in the average frequency. In this case, the frequency dispersion is negligible. On the other hand, for the nonuniform network (solid line) we see a gradual decline in synchronization accompanied by a gradual recovery in average frequency and a pronounced frequency dispersion

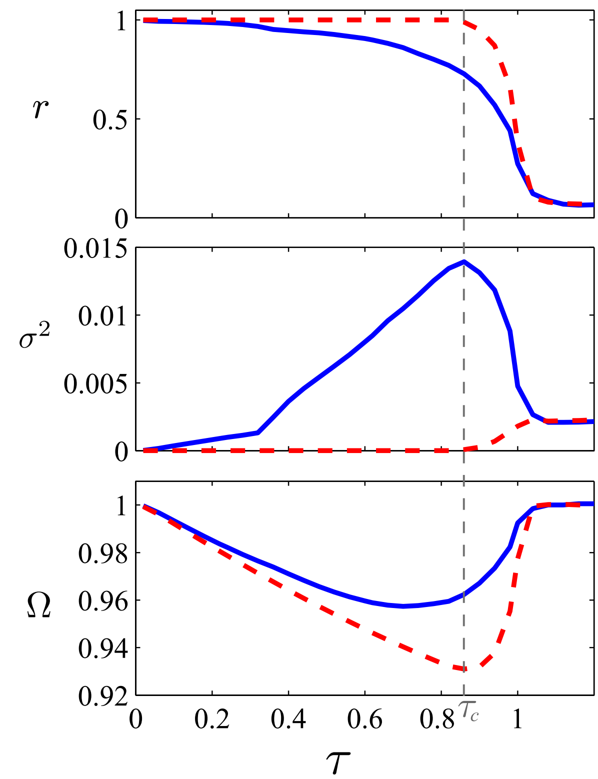

FIG. 1. (Color online) Time-averaged synchronization r, mean frequency dispersion σ^2, and mean frequency Ω as a function of time delay τ for a network of time-delayed Kuramoto oscillators with identical degree distribution (dashed lines) and binomial degree distribution (solid lines). For each degree distribution, the values presented are averages taken from computer simulations of Eqs. (2) performed on 32 different networks each consisting of a total of 150 oscillators. In the case of an identical degree distribution we see an abrupt shift from synchronization to incoherence and negligible frequency dispersion. For the binomial degree distribution we see a more gradual shift from synchronization to incoherence accompanied by a pronounced frequency dispersion. The critical time delay $\tau_c = 0.86$ marks the beginning of the transition from synchronization to incoherence for the uniform network, which happens to coincide with the maximum of the frequency dispersion for the nonuniform network. The mean node degree $\langle d_i \rangle$ was equal to 4 in all cases, other parameters values were $K = 0.1$, $k = 4$, and $\omega = 1$.

over a large time delay interval. In Sec. III we will try to explain qualitatively this difference in behavior. The critical time delay $\tau_c = 0.86$ (also depicted in Fig. 1) marks the beginning of the transition from synchronization to incoherence for the uniform network, which happens to coincide with the maximum of the frequency dispersion for the nonuniform network.

III. MEAN FIELD ANALYSIS

One of the keys to understanding the results lies in observing what happens with the average frequency as the time delay is increased. To get a feeling for this, we assume that we have a frequency-locked synchronized state given by $\phi_i(t) = \Omega t$. If we insert this expression into Eq. (2) and assume a uniform network, then we obtain the following self-consistency equation [3]

$$\Omega = \omega - K \sin(\Omega\tau). \tag{8}$$

One can verify that the frequency as a function of time delay for the uniform network presented in Fig. 1 fits Eq. (8) perfectly for time delays lower than τ_c. In order to see how this hints at an explanation for the frequency dispersion in the case of nonuniform networks, the first thing one could observe is that if $|K| < |\omega - \Omega|$ then Eq. (8) has no solution. For the nonuniform network, the degrees are not the same for all nodes, hence, due to this nonuniformity in effective coupling strength, one could anticipate a situation where the oscillators of lower degree successively lose contact with the larger cluster. In order to verify this, from the computer simulations we have calculated (see Fig. 2) the average frequency as a function of time delay for each node degree taken separately. In the figure we let Ω^d denote the average frequency of all oscillators with degree d. As we can see, on average the nodes with lower degree lose connection with the larger cluster more quickly. With the purpose of analyzing this more quantitatively, we define the effective coupling constant for each oscillator K_i as

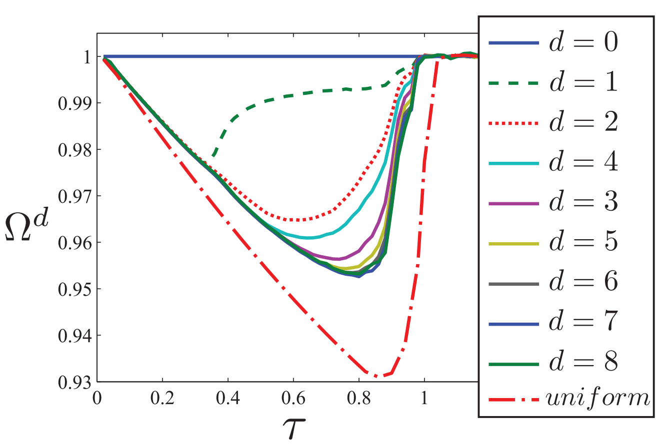

FIG. 2. (Color online) Average frequency Ω^d of all oscillators with degree d as a function of time delay on a network with binomial degree distribution. As can be seen, on average the nodes with lower degree lose connection with the larger cluster more quickly. The average frequency for the uniform network is also inserted in the figure as a reference. The network used in the simulation consisted of a total of 300 oscillators with mean node degree $\langle d_i \rangle$ equal to 4. Other parameters values were $K = 0.1$, $k = 4$, and $\omega = 1$.

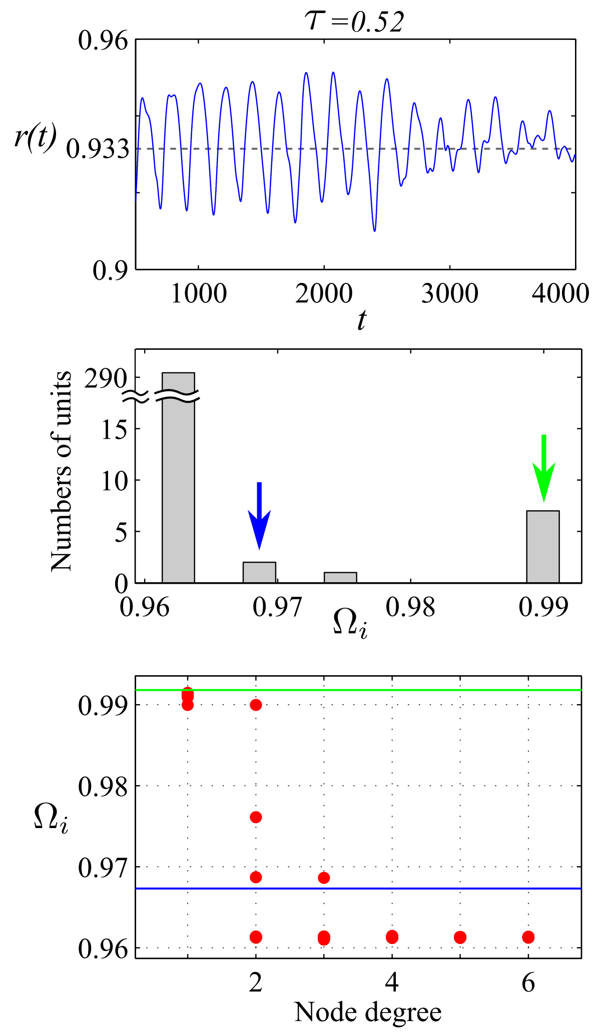

FIG. 3. (Color online) Time evolution of the synchronization order parameter $r(t)$ with root mean square $r = 0.93$ (top), histogram over the time-averaged frequencies of the oscillators (center), and frequencies versus node degree (bottom) for $\tau = 0.52$. In this case Eq. (12) explains the numerical outcome to a good approximation. The values of Eq. (12), for $d_i = 1$ and $d_i = 2$, are marked with arrows in the histogram and solid lines in the bottom figure. The result was obtained from a computer simulation of Eqs. (2) on a network with a binomial degree distribution with 300 nodes. The mean node degree $\langle d_i \rangle$ was equal to 4, other parameters values were $K = 0.1$, $k = 4$, and $\omega = 1$.

follows:

$$K_i \equiv \frac{K d_i}{k}, \tag{9}$$

where k, as usual, is the average degree of the network. Now, we consider the following differential equation:

$$\dot{\phi}_i = \omega + K_i \sin(\Omega_\tau t - \phi_i). \tag{10}$$

Here we have a system with only one oscillator interacting with a larger cluster with frequency Ω_τ (which is assumed to depend on the time delay τ). According to symbolic mathematical software, Eq. (10) has an exact solution, but it appears to be useless. Numerical simulations indicate, as it could be suspected, that if $|K_i| > |\omega - \Omega_\tau|$ then ϕ_i will be locked at

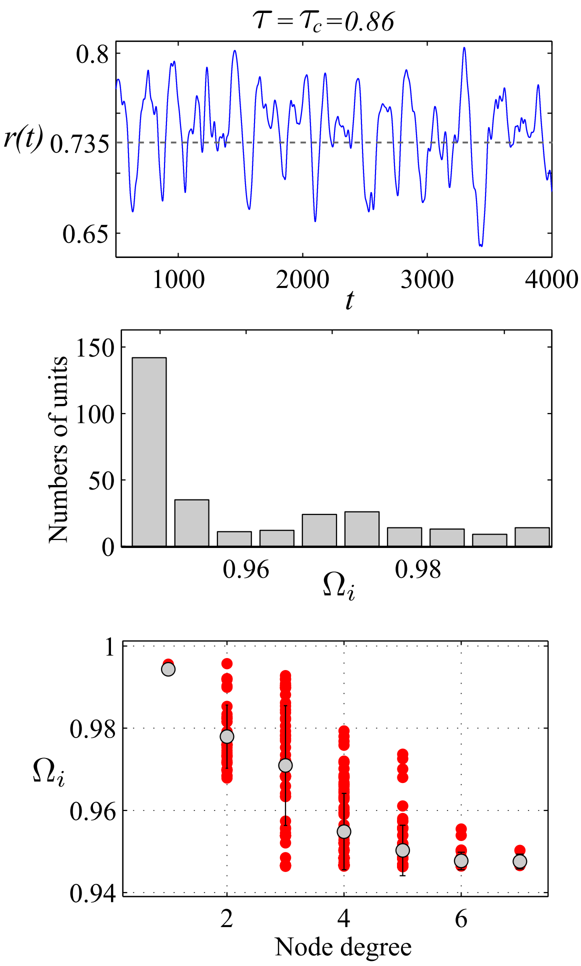

FIG. 4. (Color online) Time evolution of the synchronization order parameter $r(t)$ with root mean square $r = 0.735$ (top), histogram over the time-averaged frequencies of all the oscillators (center), and frequencies versus node degree (bottom) for $\tau = \tau_c = 0.86$. In this case approximately half of the oscillators are locked to a common frequency while the others are spread out. The gray circles in the bottom figure mark the average frequency for each node degree. The result was obtained from a computer simulation of Eqs. (2) on a network with a binomial degree distribution with 300 nodes. The mean node degree $\langle d_i \rangle$ was equal to 4, other parameters values were $K = 0.1$, $k = 4$, and $\omega = 1$.

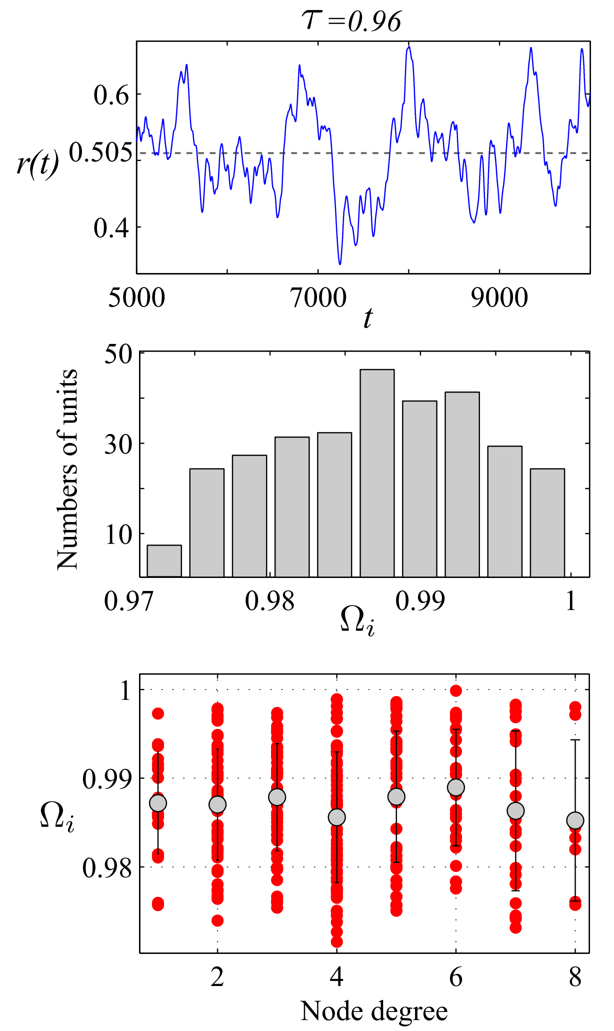

FIG. 5. (Color online) Time evolution of the synchronization order parameter $r(t)$ (top), histogram over the time-averaged frequencies of all the oscillators (center), and frequencies versus node degree (bottom) for $\tau = 0.96$. In this case the frequency distribution looks almost uniform and the order parameter fluctuates erratically. At this time delay, the time-averaged order parameter r is equal to 0.505. The gray circles in the bottom figure mark the average frequency for each node degree. The result was obtained from a computer simulation of Eqs. (2) on a network with a binomial degree distribution with 300 nodes. The mean node degree $\langle d_i \rangle$ was equal to 4, other parameters values were $K = 0.1$, $k = 4$, and $\omega = 1$.

the frequency Ω_τ. If we assume instead that $|K_i| < |\omega - \Omega_\tau|$ and treat K_i as a small parameter, then we can perform a perturbation analysis by assuming the following form of the solution:

$$\phi_i = \psi_0 + K_i \psi_1 + K_i^2 \psi_2 + \cdots . \quad (11)$$

According to such a perturbation analysis, ϕ_i will propagate with an effective time-averaged frequency ω_i which, including only the lowest-order correction term, is given by

$$\omega_i = \omega - \frac{1}{2} \frac{K_i^2}{\omega - \Omega_\tau}. \quad (12)$$

In Figs. 3, 4, and 5, we show the time evolution of the synchronization $r(t)$ as well as a histogram of the frequency of the oscillators obtained from computer simulations of the nonuniform network at three different time delays. In Fig. 3, we show the results obtained for $\tau = 0.52$, which is within the region of frequency dispersion but not close to the critical transition point. Here we can see that most oscillators are locked to a common frequency, but there are also ensembles propagating at other frequencies. In particular, two ensembles appear around the frequencies $\Omega = 0.99$ and $\Omega = 0.97$ corresponding to the values of Eq. (12) for $d_i = 1$ and $d_i = 2$ (putting Ω_τ equal to the average frequency obtained in the simulation). Hence, in this case Eq. (12) explains the numerical outcome to a good approximation. The bottom panel of Fig. 3 plots the average frequency against node degree d_i for each oscillator. It shows clearly that the oscillators with lower degree tend to have an average frequency closer to ω, while the nodes of higher degree are locked to a common frequency. The solid lines in the bottom panel mark the analytical predictions of Eq. (12) mentioned before.

In Fig. 4 the time delay has been chosen to be corresponding to the critical transition point between synchronization and incoherence for the uniform network, that is $\tau = \tau_c = 0.86$. In this case, approximately half of the oscillators are locked to a common frequency while the others are spread out. It is worth noticing that at this time delay the frequency dispersion reaches its maximum and begins to drop for larger τ. At the time delay $\tau = 0.96$, depicted in Fig. 5, we have reached the midpoint of the transition where the time-averaged order parameter r has dropped close to 0.5. In this case, the frequency distribution looks almost uniform and the order parameter $r(t)$ fluctuates erratically. At this point, it is worth mentioning that, on a uniform network, it is very hard (or perhaps impossible) to find stationary states with the order parameter $r(t)$ fluctuating around a value in the middle between 0 and 1, as has just been demonstrated on nonuniform networks. From Fig. 1 one might get the impression that these states exist also for uniform networks in a small window after the critical time delay τ_c is passed, however, the apparent continuity of the synchronization curve in the case of the uniform network comes from averaging over many different network configurations.

IV. CONCLUSIONS

In this paper we have studied sparsely connected networks of time-delayed Kuramoto oscillators with identical and binomial degree distributions respectively. The results differed in the sense that on the network with binomial degree distribution, a pronounced frequency dispersion is observed along with a relatively smooth transition from synchronization to incoherence. An explanation for this was given as a three-step cause and effect scheme. First, the time delay induces a suppression of the time-averaged frequency of the network. Due to the nonuniformity in effective coupling strength over the network with binomial degree distributions, each node has a different ability to cope with this suppression of frequency leading eventually to frequency dispersion. As a last consequence, this spread in frequency leads to time-averaged levels of synchronization intermediate between perfect synchronization and complete incoherence. We hope that this paper has thereby shed some further light on the mechanisms behind synchronization and incoherence in networks with time-delayed interactions.

ACKNOWLEDGMENTS

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