Commun Nonlinear Sci Numer Simulat 22 (2015) 1158–1168

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Signal generation and enhancement in a delayed system

J.H. Yang [a,*], Miguel A.F. Sanjuán [b], H.G. Liu [a]

[a] School of Mechatronic Engineering, China University of Mining and Technology, Xuzhou 221116, PR China
[b] Nonlinear Dynamics, Chaos and Complex Systems Group, Departamento de Física, Universidad Rey Juan Carlos, Tulipán s/n, 28933 Móstoles, Madrid, Spain

ARTICLE INFO

Article history:
Received 8 May 2014
Received in revised form 24 July 2014
Accepted 2 August 2014
Available online 14 August 2014

Keywords:
Time delay
Nonlinear resonance
Signal generation
Signal enhancement

ABSTRACT

We present an effective way to generate periodic signal in a delayed nonlinear system. The generated signal at a desired frequency can be enhanced greatly by a nonlinear term. Specifically, the amplitude of the generated signal can be much larger than that of the input signal by the cooperation of an appropriate time delay and a nonlinear term. When the primary, subharmonic or superharmonic resonance occurs, the desired signal is improved significantly. Further, the delay time that induces the resonance behavior can be analytically predicted. The result herein is different from the phenomenon of stochastic resonance or vibrational resonance observed in time delayed systems. Compared with the stochastic resonance or vibrational resonance, the method used here is much simpler and the desired signal can be generated in a much wider scope without changing the excitation. We believe that our results might be useful for the signal processing research.

1. Introduction

Signal processing research nearly involves every scientific and engineering field. From previous works, we know that a weak low-frequency signal can be enhanced significantly by an appropriate amount of noise, a high-frequency signal, or a time delay, when the response of the considered system shows a resonant behavior at the frequency of the external excitation [1–5]. In [1], the signal is enhanced by a noise, and the phenomenon is well known as stochastic resonance. In [2], the signal is enhanced by a high-frequency signal, and the phenomenon is known as vibrational resonance. Vibrational resonance is similar to stochastic resonance, but the amount of noise is replaced by a high-frequency signal. In [3], the signal is enhanced by a delayed feedback. In [4,5], the signal is enhanced by both a noise and a delay time. In [6–8], the signal is enhanced by the cooperation of the delayed feedback and a high-frequency signal. For this case, the signal is enhanced when the vibrational resonance phenomenon occurs with the help of a time delay. All these works [1–8] have studied the signal enhancement problem precisely at the frequency of the excitation signal. Furthermore, the discussions are limited to a linear response mechanism. It is because the nonlinear frequency is not considered in these works.

According to the nonlinear dynamical theory, besides the primary frequency, the resonance also occurs at the subharmonic or superharmonic frequencies [9–11]. Usually, the response amplitude at the primary, and at some special subharmonic or superharmonic frequencies can be solved approximately by different analytical methods [12–16], *e.g.*, the multiple scale method, the perturbation method, the harmonic balance method and the averaging method, *etc*. The former researches about the nonlinear response property of the system are mainly focused on some special subharmonic or superharmonic frequency, such as 1/2, 1/3, 2 or 3 times of the excitation frequency [17–20], and so on. Thus, a question arises,

* Corresponding author.
E-mail addresses: jianhuayang@cumt.edu.cn (J.H. Yang), miguel.sanjuan@urjc.es (Miguel A.F. Sanjuán).

http://dx.doi.org/10.1016/j.cnsns.2014.08.005
1007-5704/

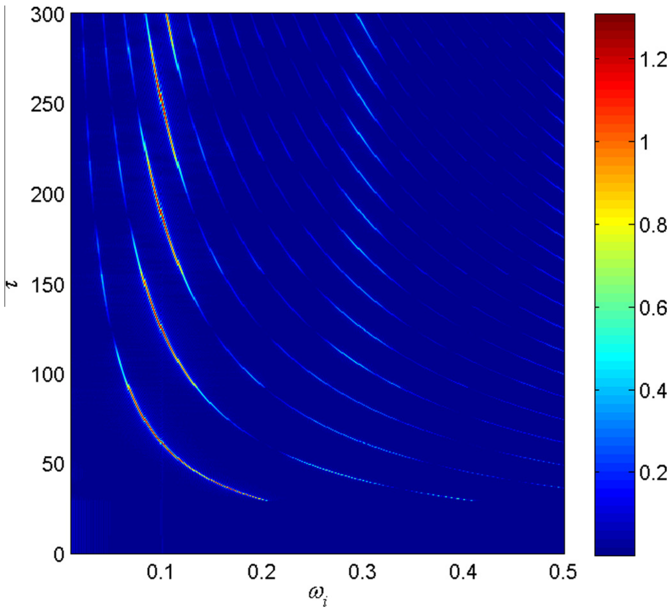

Fig. 1. Contour plot of the response amplitude $Q(\omega_i)$ in the $\tau-\omega_i$ plane. The horizontal axis is the frequency component ω_i which is given in Eq. (2). The vertical axis is the delay time τ which is included in Eq. (1). A color code plot shows the value of $Q(\omega_i)$. The signal parameters for simulation are $f = 0.1$ and $\omega = 0.1$.

whether is there any frequency component included in the response. Apparently, it is impossible. The subharmonic and the superharmonic frequency components are caused by the nonlinearity in the system. Generally speaking, there are only a few frequency components in the response and they can be obtained analytically. If we want to generate a periodic signal in a wide frequency domain, a time delay is usually introduced as an effective tool. When the delay is considered, the natural frequency of the linear system depends on the delay closely. In general terms, without any excitation, the free vibration of the delayed system has families of periodic solutions [21]. The technical problem is not to generate or enhance a signal separately, but rather to realize the two goals synchronously, and this is precisely the motivation of our research.

In this paper, we will show that the resonant behavior in fact can occur in a wide scope besides the primary, subharmonic or superharmonic frequencies that have been investigated in the previous mentioned works, leading to the generation of a steady new signal and then improving it greatly. Here, we design a simple delayed system with a cubic nonlinearity, *i.e.*,

$$\dot{x}(t) = x(t-\tau) - x^3(t) + f\cos(\omega t), \tag{1}$$

where τ is the delay time and $f\cos(\omega t)$ is a weak low-frequency external excitation. When this system is excited by both an additive noise and a harmonic signal, the signal amplification problem is investigated in the non-adiabatic stochastic resonance mechanism [22], and the delay time plays an important role in the signal amplification. In fact, the time delay is widely existent in real systems. For example, in a bursting neuron, the membrane potential of an autapse depends on a self-feedback current with time delay [23,24]. In the laser system, the light traveling from one point to another induces time delay [25]. In the machining operation, the source of time delay is due to the dependence of the cutting force on the thickness of the workpiece [13]. Time delay also exists in gene dynamics [26,27], physiological dynamics[28], population dynamics [29], the spread and the dynamics of infectious diseases [30], motor control systems [31], *etc*. As to Eq. (1), if this model is considered as a mechanism system, in which the inertia term is ignored due to it is much smaller than the damping term, the cubic term is a nonlinear spring term, and the delayed term is a controller. In this paper, we do not consider it as a mechanism system, but make it as a model for a signal generator and amplifier.

The outline of the paper is composed as follows. In Section 2, the delay induced multiple resonances in the response are introduced briefly. In Section 3, the signal enhancement problem in the resonance mechanism is investigated in detail. In Section 4, for an arbitrary delay time, the signal generation and enhancement phenomena are demonstrated. Finally, the novel results of the paper are described in the conclusions.

2. Delay-induced multiple resonances

Due to the existence of the time delay and the cubic term, the response of Eq. (1) has the form

$$x(t) = \sum_{i=1}^{n} Q(\omega_i)\cos(\omega_i t + \theta_i). \tag{2}$$

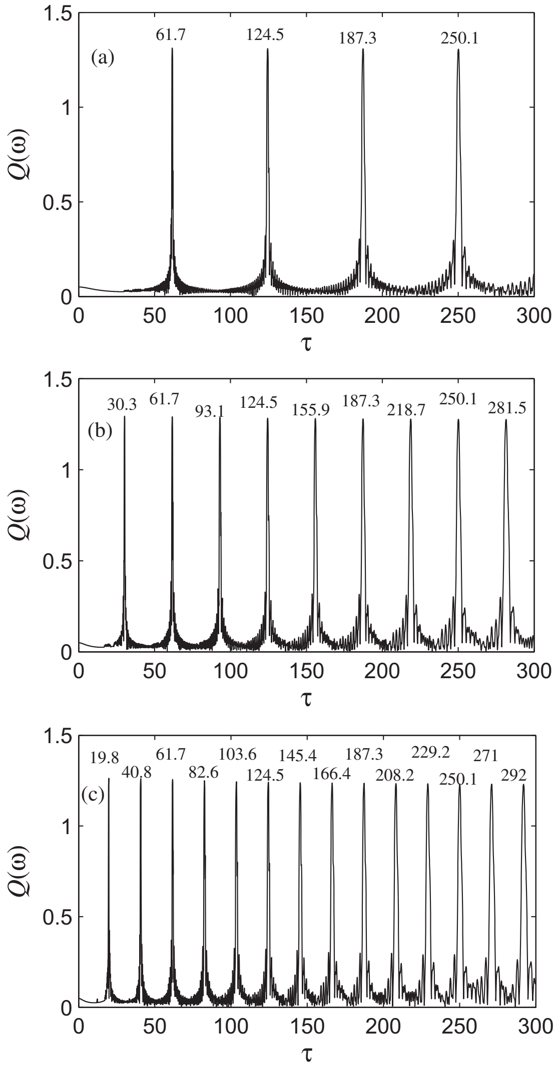

Fig. 2. Extraction of the response amplitude $Q(\omega)$ from the time series for $f = 0.1$, and $\omega = 0.1$ in (a), $\omega = 0.2$ in (b), $\omega = 0.3$ in (c).

Herein, $Q(\omega_i)$ is the amplitude of the output at the frequency ω_i, and θ_i is the corresponding initial phase angle. In Eq. (2), the response $x(t)$ contains the primary frequency $\omega_i = \omega$, the subharmonic frequency $\omega_i < \omega$ and the superharmonic frequency $\omega_i > \omega$. The frequency component ω_i may be induced by the excitation frequency ω, the nonlinearity x^3, and the natural frequency of the linear delayed system which closely depends on the time delay τ. Here, we denote the natural frequency of the corresponding linear delayed system as ω_n. According to the vibration theory [18], the primary resonance occurs when the excitation frequency approaches to the natural frequency, *i.e.*, $\omega = \omega_n + \epsilon\sigma$. Here, σ is the detuning parameter and ϵ is a small parameter, so that $\epsilon\sigma$ is a small quantity. Moreover, the subharmonic or superharmonic resonance occurs when the natural frequency and the excitation frequency satisfy $\omega = m\omega_n + \epsilon\sigma$, where m is a positive number. We have the subharmonic resonance when $m > 1$, and the superharmonic resonance when $0 < m < 1$.

Although Eq. (1) is in a simple form, it is difficult to obtain the analytical expression of the solution. In the following analysis, we mainly discuss the amplitude of the output according to numerical simulations. The amplitude of the output at the frequency ω_i which denoted by $Q(\omega_i)$ is numerically calculated through

$$Q(\omega_i) = \sqrt{Q_{\sin}^2(\omega_i) + Q_{\cos}^2(\omega_i)}, \tag{3}$$

where $Q_{\sin}(\omega_i)$ and $Q_{\cos}(\omega_i)$ are the Fourier coefficients computed by

$$Q_{\sin}(\omega_i) = \frac{2}{rT}\int_0^{rT} x(t)\sin(\omega_i t)dt, \quad Q_{\cos}(\omega_i) = \frac{2}{rT}\int_0^{rT} x(t)\cos(\omega_i t)dt, \tag{4}$$

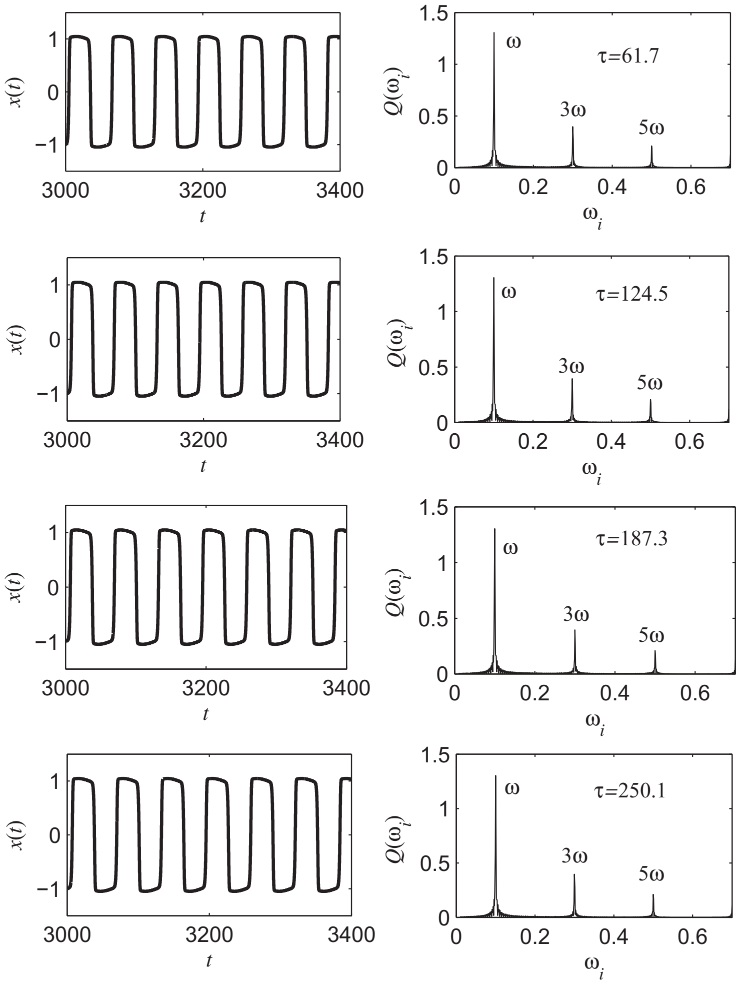

Fig. 3. Truncated steady time series (on the left column) and the corresponding response amplitude $Q(\omega_i)$ at the frequency ω_i (on the right column) for $f = 0.1, \omega = 0.1$ and τ =61.7, 124.5, 187.3, 250.1 from top to bottom.

herein $T = 2\pi/\omega, r$ is a positive integer which should be large enough, and $x(t)$ is the time series which is directly calculated from the original equation. We name $Q(\omega_i)$ as the response amplitude at the frequency ω_i. If $Q(\omega_i) > f$ with $\omega_i = \omega$, the input signal is enhance by the system. If $Q(\omega_i) > f$ with $\omega_i \neq \omega$, a new signal is generated and enhanced by the system. In our numerical simulations, the total time is 200T. After removing the first 100T as the transient response, we make the last 100T as the steady time series. Eq. (1) is discretized by using a simple Euler scheme, *i.e.*,

$$x(n+1) = x(n) + [x(n - \tau/\Delta t) - x(n)^3 + f\cos(\omega n \Delta t)]\Delta t. \tag{5}$$

We choose $\Delta t = 0.01$ as the time step. When the delay time τ is a controllable parameter, we use $\Delta\tau = 0.1$ as the delay time step.

In order to get a global view, according to Eqs. (2) and (3), the effect of the delay time τ on the response amplitude $Q(\omega_i)$ at the frequency ω_i is illustrated in a two-dimensional plane, as shown in Fig. 1, where a color code plot of the value of $Q(\omega_i)$ appears. In this figure, some light lines are separated by dark areas, which indicates the occurrence of the multiple resonances. On the one hand, for a fixed delay time τ, there may exist a primary resonance frequency and some other subharmonic or superharmonic resonance frequencies. Apparently, the resonance frequency may be equal to the frequency of the input, or may be smaller or larger than it. It depends on the delay time completely. On the other hand, for a fixed frequency, we make the resonance occur at this frequency by varying the delay time. To reveal the signal generation and enhancement problem further, we will analyze different resonance patterns in the following discussions.

3. Signal enhancement in the resonance mechanism

In this section, we will discuss the signal enhancement problem based on the resonance mechanism. Throughout this section, we will see that the input signal will be enhanced when the primary resonance occurs, and the generated new signal

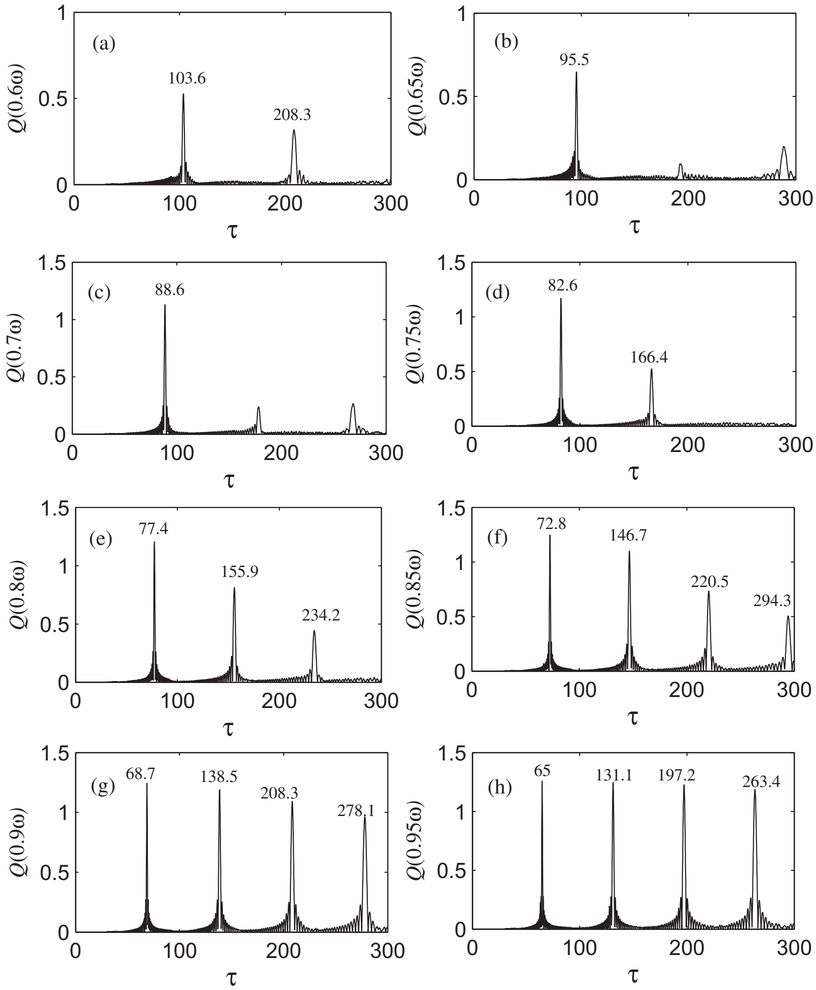

Fig. 4. Extraction of the response amplitude $Q(\omega_i)$ from the time series for $f = 0.1, \omega = 0.1$, and $\omega_i = 0.6\omega, 0.65\omega, 0.7\omega, 0.75\omega, 0.8\omega, 0.85\omega, 0.9\omega, 0.95\omega$ from (a) to (h).

will be enhanced when the subharmonic or superharmonic resonance occurs. The time delay has an indelible contribution to the signal generation and signal enhancement.

3.1. Signal enhancement by the primary resonance

According to Eq. (3), we first calculate the response amplitude at the excitation frequency ω. With the increase of the delay time τ, the multiple primary resonance is clearly shown in Fig. 2. For the case $\omega = 0.1$, the response presents a strong resonance at the delay values τ = 61.7, 124.5, 187.3 and 250.1 in Fig. 2(a). For the case $\omega = 0.2$, the resonance occurs at $\tau =$ 30.3, 61.7, 93.1, 124.5, 155.9, 187.3, 218.7, 250.1 and 281.5 in Fig. 2(b). For the case $\omega = 0.3$, the resonance occurs at $\tau = 19.8$, 40.8, 61.7 82.6, 103.6, 124.5, 145.4, 166.4, 187.3, 208.2, 229.2, 250.1, 271 and 292 in Fig. 2(c). We take the resonance as a strong one is because the response amplitude of the output at the resonance peak is much larger than the amplitude of the input signal. Interestingly, we realize it only by a delay time, and it does appear without the aid of a high-frequency signal or a noise. Hence, this shows that there is another way to enhance the weak low-frequency signal besides the stochastic resonance [1] and the vibrational resonance [2,6–8] mechanisms. Moreover, the level of the signal enhancement induced by the delay time is really high.

Also in Fig. 2, the resonance occurs at some specific values. Further, these values can be predicted approximately by the analytical treatment. By ignoring the nonlinearity and the harmonic excitation in Eq. (1), we only consider the linear delayed system, *i.e.*,

$$\dot{x}(t) = x(t - \tau). \tag{6}$$

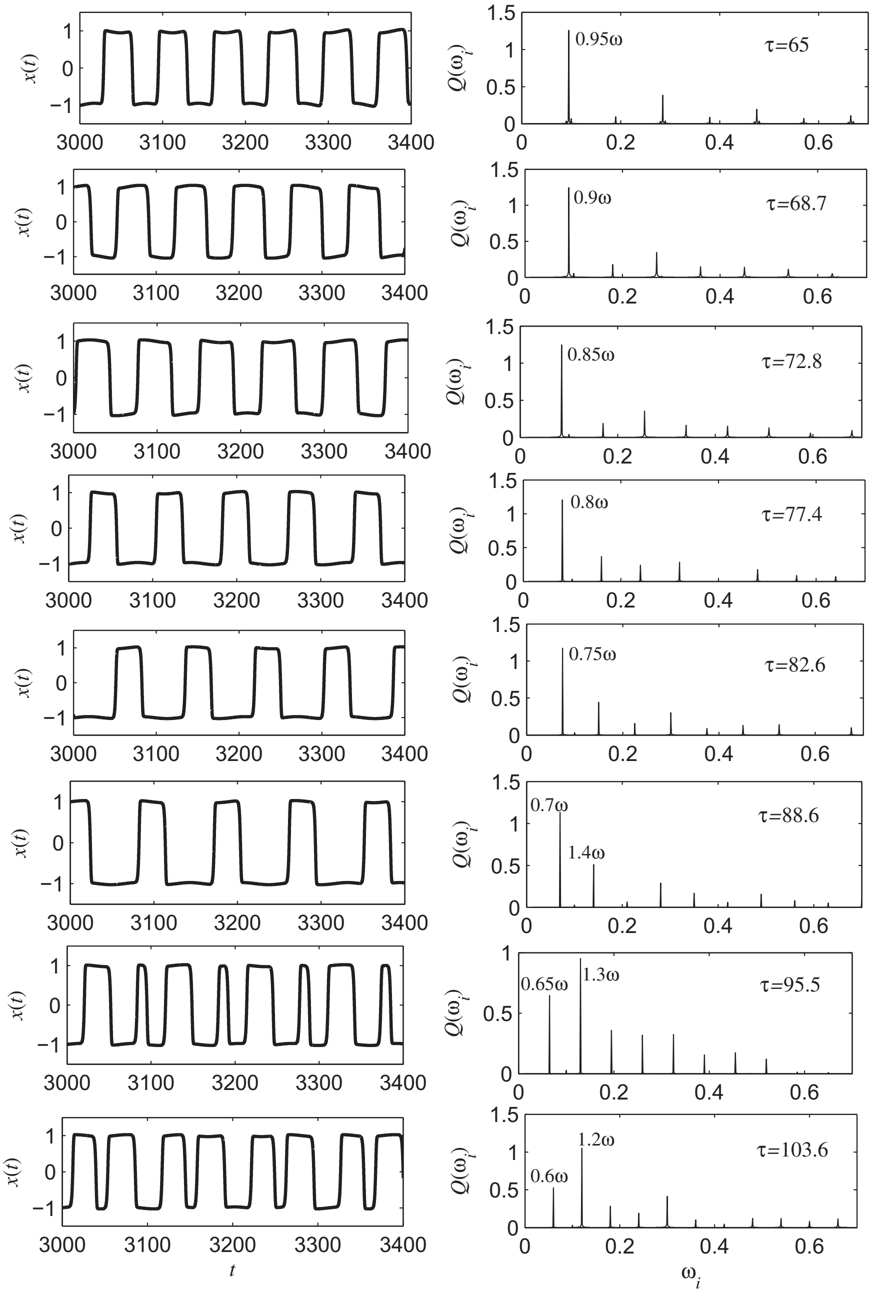

Fig. 5. Truncated steady time series (on the left column) and the corresponding response amplitude $Q(\omega_i)$ at the frequency ω_i (on the right column) for $f = 0.1, \omega = 0.1$ and $\tau = 65, 68.7, 72.8, 77.4, 82.6, 88.6, 95.5, 103.6$ from top to bottom.

According to the linear vibration theory, the imaginary parts of the eigenvalues of Eq. (6) is determined by the system's natural frequency ω_n. When the response is in periodic motion, we search the solution of Eq. (6) in the complex form

$$x = Qe^{(i\omega_n t)}. \tag{7}$$

Substituting Eq. (7) into Eq. (6) and separating the imaginary parts from equation, then one has

$$\omega_n = -\sin(\omega_n \tau). \tag{8}$$

Further solving Eq. (8), one obtains

$$\tau = \frac{1}{\omega_n}(2p\pi - \sin^{-1}\omega_n), \tag{9}$$

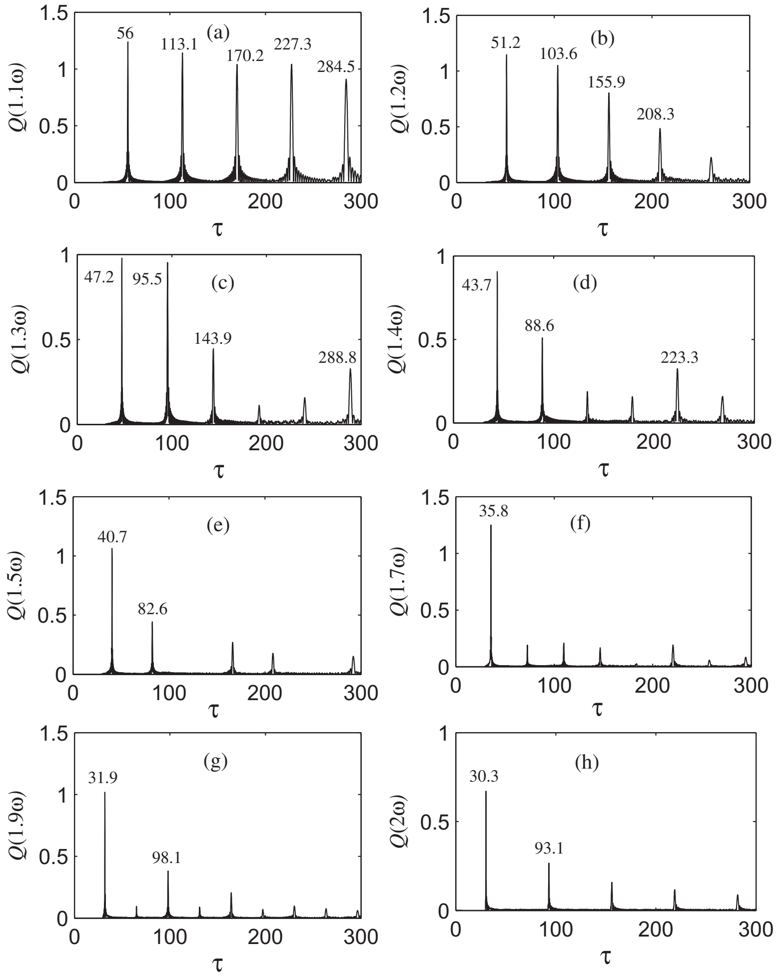

Fig. 6. Extraction of the response amplitude $Q(\omega_i)$ from time series for $f = 0.1$, $\omega = 0.1$, and $\omega_i = 1.1\omega, 1.2\omega, 1.3\omega, 1.4\omega, 1.5\omega, 1.7\omega, 1.9\omega$ and 2ω from (a) to (h).

where p is a positive integer number. Certainly, $\omega_n \leqslant 1$ in Eq. (9), this condition is satisfied when the external excitation is a low-frequency signal. The primary resonance occurs when $\omega = \omega_n$. According to Eq. (9), in the range $\tau \in [0, 300]$, we obtain the analytical results of the delay time for the primary resonance are $\tau \approx 61.8, 124.7, 187.5, 250.3$ for the case $\omega = 0.1$, and $\tau \approx 30.4, 61.8, 93.2, 124.7, 156.1, 187.5, 218.9, 250.3, 281.7$ for the case $\omega = 0.2$, and $\tau \approx 19.9, 40.9, 61.8, 82.8, 103.7, 124.6, 145.6, 166.5, 187.5, 208.4, 229.4, 250.3, 271.3, 292.2$ for the case $\omega = 0.3$. Thus, it can be seen that the analytical values of τ which induce the primary resonance agree with the corresponding numerical results that shown in Fig. 2. The small error may arise from the nonlinearity and the numerical integration. The error is located within the range allowed. Moreover, usually in engineering, the interest lies in the bandwidth of resonance peak. The bandwidth is defined by the cutoff frequencies, at which $Q = \frac{Q_{max}}{\sqrt{2}}$ [32]. According to Eq. (9) again, we are also able to predict the subsequent resonant peaks beyond the delay scope in Fig. 2. Specifically, if we want to obtain the location of the $p - th$ resonance peak, we only need to substitute the positive integer p into Eq. (9). Another fact from both the analytical and numerical results is the periodic property, of period $2\pi/\omega$, of Q versus the delay time τ. The signal can be enhanced effectively due to the cooperation of the delay time and the nonlinearity term. Without the negative cubic term in the equation, the response would diverge easily.

In order to explain the delay-induced multiple resonances further, corresponding to some delay values that can induce resonance peaks in Fig. 2, the truncated steady time series and the response amplitudes $Q(\omega_i)$ of the output at the frequency ω_i are shown in Fig. 3. On the one hand, when τ = 61.7, 124.5, 187.3 and 250.1, the time series or the response amplitude $Q(\omega_i)$ at the frequency ω presents obvious periods in each subplot. The figure displays strong resonance peaks at the frequency ω and weak resonance peaks at the frequency 3ω and 5ω. It is because the specific τ and $3\omega, 5\omega$ still satisfy Eq. (9) approximately in the subplot. In other words, there are many natural frequencies for a delay fixed system. These frequencies are called the first-, second-,...pth-order natural frequency respectively, when ordered sequentially. On the other hand, the time series and the curves $Q(\omega_i) - \omega_i$ in these subplots are almost identical. It verifies the periodic property of the

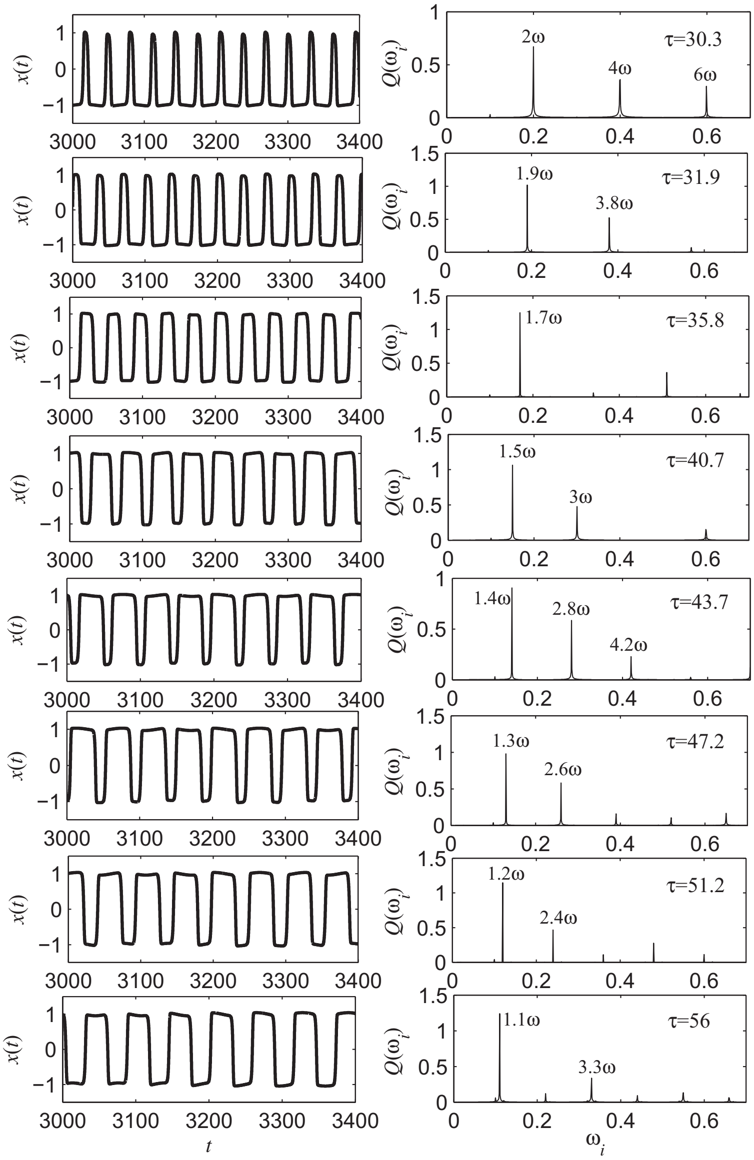

Fig. 7. Truncated steady time series (on the left column) and the corresponding response amplitude $Q(\omega_i)$ at the frequency ω_i (on the right column) for $f = 0.1, \omega = 0.1$ and $\tau = 30.3, 31.9, 35.8, 40.7, 43.7, 47.2, 51.2, 56$ from top to bottom.

response versus the delay time. From the analysis in this section, we believe that the nonlinear delayed system is an effective tool in enhancing a weak signal.

3.2. Signal enhancement by the subharmonic resonance

If we need the system to generate a signal with the frequency smaller than that of the input signal, we realize it by choosing an appropriate delay. When the subharmonic resonance occurs, the desired signal gives birth. We extract the response amplitude $Q(\omega_i)$ at the specific frequency ω_i from the time series, as is shown in Fig. 4. We choose ω_i = $0.6\omega, 0.65\omega, 0.7\omega, 0.75\omega, 0.8\omega, 0.85\omega, 0.9\omega$ and 0.95ω. Through varying the delay time τ, we make the resonance occur at these frequencies. As a result, a signal with the subharmonic frequency $\omega_i < \omega$ is produced according to our demand. In this figure, the delay time corresponding to the first resonance peak turns smaller when the desired frequency increases from 0.6ω to 0.95ω. Specifically, these numerical values are τ = 103.6, 95.5, 88.6, 82.6, 77.4, 72.8, 68.7 and 65, as is labeled in Fig. 4. Moreover, compared with these resonance phenomena, there are more and stronger resonance peaks when the resonance occurs near the excitation frequency. Similar to the analysis in the previous subsection, we give an explanation as to why these resonances occur at some specific delay values. If we substitute $p = 1$ and $\omega_n = 0.6\omega, 0.65\omega, 0.7\omega, 0.75\omega, 0.8\omega, 0.85\omega, 0.9\omega$ and

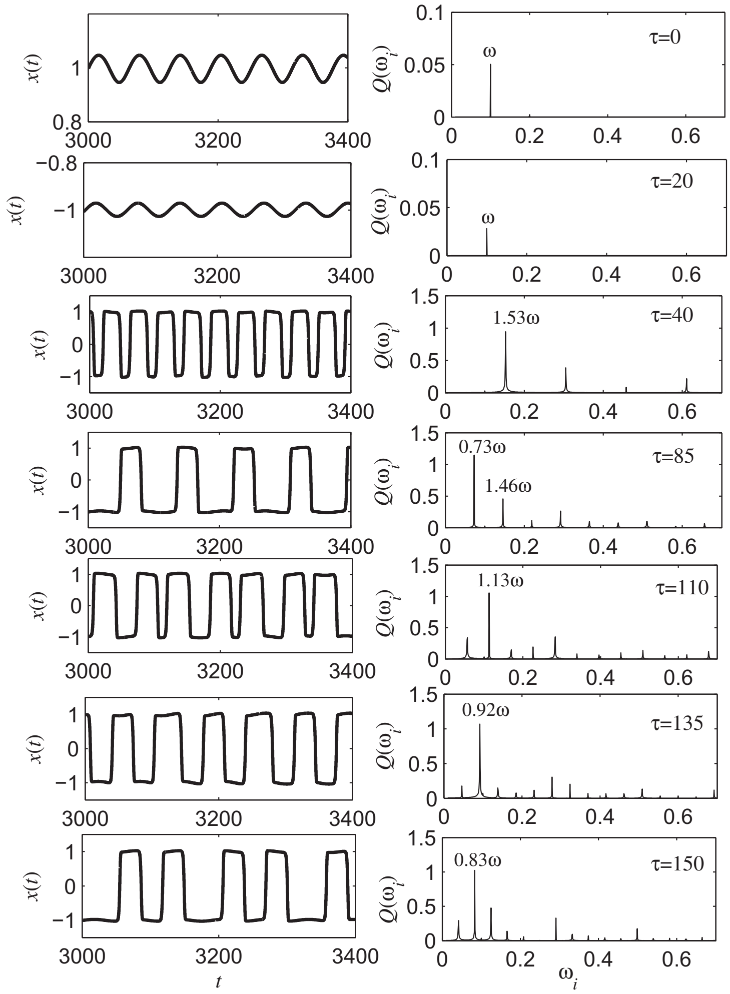

Fig. 8. Truncated steady time series (on the left column) and the corresponding response amplitude $Q(\omega_i)$ at the frequency ω_i (on the right column) for $f = 0.1, \omega = 0.1$ and $\tau = 0$, 20, 40, 85, 110, 135, 150 from top to bottom.

0.95ω into Eq. (9), then we obtain the delay time corresponding to the first resonant peak is $\tau \approx 103.7$, 95.7, 88.8, 82.8, 77.5, 72.9, 68.8 and 65.1 respectively. The analytical results agree approximately with the numerical calculations.

At the delay time that induces the first resonance peak in each subplot in Fig. 4, the truncated steady time series and the corresponding response amplitude $Q(\omega_i)$ at the frequency ω_i are plotted in Fig. 5. When $\tau = 65$, 68.7, 72.8, 77.4, 82.6 and 88.6, the response amplitude reaches the strong resonance at the frequency $0.95\omega, 0.9\omega, 0.85\omega, 0.8\omega, 0.75\omega$ and 0.7ω respectively. It is coincident with the fact in Fig. 4. However, when $\tau = 95.5$, the first strong resonance occurs at the frequency 1.3ω and the second strong resonance occurs at the frequency 0.65ω. When $\tau = 103.6$, the first strong resonance occurs at the frequency 1.2ω and the second strong resonance occurs at the frequency 0.6ω. It is because the cases $\tau = 95.5, \omega_n = 1.3\omega$ and $\tau = 103.6, \omega_n = 1.2\omega$ also satisfy Eq. (9). Moreover, compared with the first-order natural frequencies $\omega_n = 0.6\omega$ and $\omega_n = 0.65\omega$, the second-order natural frequencies $\omega_n = 1.2\omega$ and $\omega_n = 1.3\omega$ are much closer to the excitation frequency ω. Hence, for the case $\tau = 95.5$, the first strong resonance peak occurs at the second-order natural frequency 1.3ω, but not at the first-order natural frequency 0.65ω. Likewise, for the case $\tau = 103.6$, the first strong resonance peak occurs at the second-order natural frequency 1.2ω, but not at the first-order natural frequency 0.6ω. For the same reason, the strongest resonance peak turns bigger gradually from Fig. 4(a) to Fig. 4(h).

3.3. Signal enhancement by the superharmonic resonance

The strong resonance not only occurs at the primary and subharmonic frequencies, but also occurs at the superharmonic frequency. With the increase of the delay time τ, we calculate the response amplitude $Q(\omega_i)$ at the desired frequency $\omega_i = 1.1\omega, 1.2\omega, 1.3\omega, 1.4\omega, 1.5\omega, 1.7\omega, 1.9\omega$ and 2ω respectively, as shown in Fig. 6. Reading directly from this figure, the numerical values of the delay time that induces the first resonance peak in each subplot is $\tau = 56, 51.2, 47.2, 43.7, 40.7, 35.8, 31.9$ and 30.3 respectively. According to Eq. (9), when $p = 1$, the corresponding analytical delay value that induces the first resonance at the desired frequency in each subplot is $\tau \approx 56.1, 51.4, 47.3, 43.9, 40.9, 36, 32$ and 30.4. In this figure, near the excitation frequency ω, there are more and stronger resonance peaks.

Corresponding to the first peak in each subplot in Fig. 6, we give the truncated steady time series and the response amplitude $Q(\omega_i)$ versus the frequency component ω_i in Fig. 7. All the strongest resonance peaks occur at the first-order natural frequency, since the first-order natural frequency approaches to the excitation frequency.

4. Resonance in the fixed time delay system

In Fig. 8, for some fixed delay values, the steady time series and the resonance frequencies are shown clearly. For τ = 0 and 20, there is no apparent resonance behavior. For $\tau = 40, 85, 110, 135$ and 150, the first-order natural frequency is $1.53\omega, 0.73\omega, 1.13\omega, 0.92\omega$ and 0.83ω respectively. Certainly, these values satisfy Eq. (9) approximately. It indicates the fact again that a fixed time delay can generate a new signal that is different from the input signal. The generated signal is in fact much stronger than the excitation one, and this is the result of the nonlinear response.

5. Conclusions

To conclude, we have introduced a delayed system as a signal generator to produce a signal according to our demand. Moreover, the generated signal can be enhanced in a great degree. Specifically, on the one hand, if we want to obtain a strong signal at a desired frequency, we only need to adjust the delay time in the considered system. Further, the optimal delay time that produces the desired signal can be analytically predicted. On the other hand, for the system with a fixed delay time, the response of the system has a strong resonant behavior at a natural frequency. Throughout this paper, we have shown that the signal can be generated effectively even though the signal is not included in the excitation. Furthermore, it is important to make it clear that the new signal is much stronger than the input signal. The cubic nonlinearity is a key factor to enhance the generated signal. Without this term, the response of the system is in divergent or oscillating with a small amplitude. As a result, the signal is generated and enhanced due to the cooperation of the time delay and the cubic nonlinearity. Given that the noise or the high-frequency signal is not included in the excitation, the results in this paper are different from the stochastic resonance or vibrational resonance in the delayed system. Moreover, using the method shown here to generate and improve the signal strength has much more advantages as compared to the stochastic resonance or vibrational resonance phenomena. However, the noise is unavoidable in many physical backgrounds. The system may present much more complicated dynamics especially when the nonlinear frequency is considered. It is our future work. Nevertheless, we believe that the results in this paper might have important impact on the signal processing issue.

Acknowledgements

We acknowledge financial supports by the National Natural Science Foundation of China (Grant No. 51305441), the China Postdoctoral Science Foundation (Grant No. 2012M510192), the Priority Academic Program Development of Jiangsu Higher Education Institutions, and the Spanish Ministry of Science and Innovation (Grant No. FIS2009-09898).

References

[1] Gammaitoni L, Hänggi P, Jung P, Marchesoni F. Stochastic resonance. Rev Mod Phys 1998;70:223.
[2] Landa PS, McClintock PVE. Vibrational resonance. J Phys A: Math Gen 2000;33:L433.
[3] Ravichandran V, Chinnathambi V, Rajasekar S. Nonlinear resonance in duffing oscillator with fixed and integrative time-delayed feedbacks. Pramana 2012;78:347–60.
[4] Kim S, Park SH, Pyo HB. Stochastic resonance in coupled oscillator systems with time delay. Phys Rev Lett 1999;82:1620.
[5] Wu D, Zhu S, Luo X. Cooperative effects of random time delays and small-world topologies on diversity-induced resonance. Europhys Lett 2009;86:50002.
[6] Yang JH, Liu XB. Delay induces quasi-periodic vibrational resonance. J Phys A: Math Theor 2010;43:122001.
[7] Yang JH, Zhu H. Bifurcation and resonance induced by fractional-order damping and time delay feedback in a duffing system. Commun Nonlinear Sci Numer Simulat 2013;18:1316–26.
[8] Jeevarathinam C, Rajasekar S, Sanjuan MAF. Theory and numerics of vibrational resonance in duffing oscillators with time-delayed feedback. Phys Rev E 2011;83:066205.
[9] El-Bassiouny AF, El-Kholy S. Resonances of a nonlinear SDOF system with time-delay in linear feedback control. Phys Scr 2010;81:015007.
[10] Hu H, Dowell EH, Virgin LN. Resonances of a harmonically forced duffing oscillator with time delay state feedback. Nonlinear Dyn 1998;15:311–27.
[11] Ghosh S, Ray DS. Nonlinear vibrational resonance. Phys Rev E 2013;88:042904.
[12] Xu J, Chung KW, Chan CL. An efficient method for studying weak resonant double Hopf bifurcation in nonlinear systems with delayed feedbacks. SIAM J Appl Dyn Syst 2007;6:29–60.
[13] Sun JQ, Ding Q. Advances in analysis and control of time-delayed dynamical systems. New Jersey: World Scientific; 2013.
[14] Hu HY, Wang ZH. Dynamics of controlled mechanical systems with delayed feedback. Berlin: Springer-Verlag; 2002.
[15] Jin Y, Hu H. Dynamics of a duffing oscillator with two time delays in feedback control under narrow-band random excitation. ASME J Comput Nonlinear Dyn 2008;3:021205.
[16] Thomsen JJ. Vibrations and stability: advanced theory, analysis, and tools. Berlin: Springer; 2003.
[17] Li FM, Yao G. 1/3 Subharmonic resonance of a nonlinear composite laminated cylindrical shell in subsonic air flow. Compos Struct 2013;100:249–56.
[18] Nayfeh AH, Mook DT. Nonlinear oscillations. John Wiley Sons; 2008.

[19] Kacem N, Baguet S, Dufour R, Hentz S. Stability control of nonlinear micromechanical resonators under simultaneous primary and superharmonic resonances. Appl Phys Lett 2011;98:193507.
[20] Leung AYT, Guo Z. Resonance response of a simply supported rotor-magnetic bearing system by harmonic balance. Int J Bifurcation Chaos 2012;22:1250136.
[21] Yanchuk S, Perlikowski P. Delay and periodicity. Phys Rev E 2009;79:046221.
[22] Du LC, Mei DC. Amplification of weak signals via the non-adiabatic regime of stochastic resonance in a bistable dynamical system with time delay. Phys Scr 2011;84:015003.
[23] Wang H, Ma J, Chen Y, Chen Y. Effect of an autapse on the firing pattern transition in a bursting neuron. Commun Nonlinear Sci Numer Simulat 2014;19:3242–54.
[24] Qin HX, Ma J, Jin WY, Wang CN. Dynamics of electric activities in neuron and neurons of network induced by autapses. Sci China Technol Sc 2014;57:936–46.
[25] Wirkus S, Rand R. The dynamics of two coupled van der Pol oscillators with delay coupling. Nonlinear Dyn 2002;30:205–21.
[26] Verdugo A, Rand R. Hopf bifurcation in a DDE model of gene expression. Commun Nonlinear Sci Numer Simulat 2008;13:235–42.
[27] Verdugo A, Rand R. Center manifold analysis of a DDE model of gene expression. Commun Nonlinear Sci Numer Simulat 2008;13:1112–20.
[28] Camacho E, Rand R, Howland H. Dynamics of two van der Pol oscillators coupled via a bath. Int J Solids Struct 2004;41:2133–43.
[29] Wu F, Xu Y. Stochastic Lotka–Volterra population dynamics with infinite delay. SIAM J Appl Math 2009;70:641–57.
[30] Culshaw RV, Ruan S, Webb G. A mathematical model of cell-to-cell spread of HIV-1 that includes a time delay. J Math Biol 2003;46:425–44.
[31] Liu GP, Mu JX, Rees D, Chai SC. Design and stability analysis of networked control systems with random communication time delay using the modified MPC. Int J Control 2006;79:288–97.
[32] Balachandran B, Magrab E. Vibrations. Cengage Learning; 2008.