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Vibrational subharmonic and superharmonic resonances

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ABSTRACT

We investigate the response to the low-frequency of a bistable system perturbed by a low-frequency and a high-frequency excitation. The resonance induced by the high-frequency excitation or simply by a variation of a system parameter can occur at the frequency which is equal to, or smaller, or larger than the low-frequency excitation. Similarly to what happens with the subharmonic resonance and the superharmonic resonance phenomenon, we name the resonance that occur at these frequencies as vibrational subharmonic resonance and vibrational superharmonic resonance, respectively. While the traditional vibrational resonance reported in the literature appears in a continuous region in a three-dimensional surface, however the vibrational subharmonic resonance and the vibrational superharmonic resonance reported here occur only in some discrete regions. Here, we show the new results describing both the vibrational subharmonic resonance and the vibrational superharmonic resonance at a frequency which is not an integer multiple of the low-frequency. It is different from the nonlinear vibrational resonance reported in the former literatures, which only give the high-order vibrational resonance at frequencies that are multiple of the excitation frequency.

1. Introduction

It is well known that many physical systems can be optimized by using a high-frequency excitation and its applications are numerous in many scientific and engineering fields. For example, the high-frequency excitation in a nonlinear system may have some effects such as stiffening, biasing, smoothening, *etc.* [1]. When we have a nonlinear system that is perturbed either externally or parametrically by using only a high-frequency excitation, the oscillation of the system can be enlarged at the natural frequency [2–4], for the case that the excitation frequency is far larger than the natural frequency of the corresponding linear system. When the nonlinear system is excited by both a low-frequency and a high-frequency excitation, then the high-frequency excitation can induce a resonance at the low-frequency of the weak excitation. In other words, the response of the system to the low-frequency excitation can be greatly enhanced by the high-frequency excitation. This phenomenon was named vibrational resonance (VR) in a paper published by Landa and McClintock in the year 2000 [5]. In the last years, VR has been investigated in many disciplines due to the importance of the two-frequency excitations existing in many different fields [6–11]. Most of the work done on VR has been focused on the resonance at the low-frequency excitation.

Recently, Ghosh and Ray [12] and Chizhevsky [13] found that VR can also occur at the high-order frequencies. Specifically, if the low-frequency of one excitation is ω, the vibrational resonance induced by the high-frequency excitation may occur at

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frequencies which are multiple of the frequency ω. In general, the excitation frequency ω is named as primary frequency, and the frequency $n\omega/m$ is named as superharmonic frequency and subharmonic frequency when $n/m > 1$ and $0 < n/m < 1$, respectively. When the VR occurs at the primary frequency ω, we have the traditional VR phenomenon. When the VR occurs at the superharmonic frequency $n\omega$, where $n > 1$ is a positive integer number, this kind of VR was named nonlinear VR by Ghosh and Ray [12], or vibrational higher-order resonance by Chizhevsky [13]. From theoretical considerations, a resonance can not only occur at the primary frequency ω or the superharmonic frequency $n\omega$ (n is a positive integer), but it may also occur at the frequencies $n\omega/m$. Here, n/m is positive but not limited to an integer. Thus, a question arises, whether the high-frequency excitation can induce a resonance at the subharmonic frequency $n\omega/m$ $(0 < m/n < 1)$ or at the general superharmonic frequency $n\omega/m$ ($n/m > 1$, but it is not an integer number). As a consequence, we consider it worth to analyse the resonance phenomena at the general frequencies $n\omega/m$. Among other reasons, these frequencies appear in many engineering structures and usually as a source inducing a fault in the working [14–16]. Thus, the main motivation of this work is to investigate the resonances induced by a high-frequency excitation in the framework of the nonlinear frequency response theory. In this paper, we name VR at the subharmonic frequency as the *vibrational subharmonic resonance* (VSubR) and name VR at the superharmonic resonance as the *vibrational superharmonic resonance* (VSupR). This is because the resonance is a subharmonic resonance and a superharmonic resonance at first and these resonant behaviors are mainly induced by the vibrational high-frequency perturbation then. In previous references, the subharmonic resonance and the superharmonic resonance are usually induced by only one excitation. As far as we know, it has not been reported that this kind of resonance can be enhanced by another high-frequency excitation.

Under the two harmonic excitations, the normal form of the overdamped bistable system can be written as

$$\frac{dx(t)}{dt} = ax(t) - bx^3(t) + f\cos(\omega t) + F\cos(\Omega t), \tag{1}$$

where $f\cos(\omega t)$ is the weak low-frequency excitation and $F\cos(\Omega t)$ is the high-frequency excitation, $f \ll 1$ and $\omega \ll \Omega$. The parameters a and b are positive coefficients of the linear and nonlinear terms, respectively. The paper is organized as follows. In Section 2, we will predict the potential frequency components in the response via the nonlinear frequency response theory. In Section 3, we discuss the VSubR at the frequencies $\omega/3$ and $\omega/2$, respectively. In Section 4, we discuss the VSupR at the frequencies $4\omega/3$ and $5\omega/2$ with some detail. Finally, the last section is devoted to some discussions and the main results of this work.

2. Predicting the subharmonic and superharmonic frequencies

In a linear system excited by a harmonic excitation with a given frequency, the system will respond at the very same frequency with a certain magnitude and a certain phase angle relative to the input. This is the linear frequency response theory. In a nonlinear system that is also excited by a given harmonic excitation, only the excitation frequency cannot reveal the complexity of the response. Besides the driving frequency, we also need to consider the subharmonic and superharmonic frequencies. These frequencies are induced by the nonlinear terms in the system equation. This is the nonlinear frequency response theory. The subharmonic and superharmonic frequencies in the response of Eq. (1) can be predicted by a simple analysis that follows.

According to the well-known method of direct separation of slow and fast motions [1,2] an approximate solution of Eq. (1) can be found in the form $x = X + \Psi$, where X is a slow variable and Ψ is a fast variable. In general, X is regarded as the slow motion with frequency ω and Ψ with frequency Ω. Here, we consider that X, as a slow variable, has many other low-frequencies besides the frequency ω. No matter how many frequency components X contains, as long as X is much slower than Ψ, this method still can be used. Substituting $x = X + \Psi$ into Eq. (1), one obtains

$$\frac{dX}{dt} + \frac{d\Psi}{dt} = a(X + \Psi) - b(X^3 + 3X^2\Psi + 3X\Psi^2 + \Psi^3) + f\cos(\omega t) + F\cos(\Omega t). \tag{2}$$

Solving the approximate solution of Ψ in the linear equation

$$\frac{d\Psi}{dt} = a\Psi + F\cos(\Omega t), \tag{3}$$

it is easy to obtain

$$\Psi = \frac{F}{\sqrt{\Omega^2 + a^2}}\cos\left(\Omega t + \tan^{-1}\frac{\Omega}{a}\right). \tag{4}$$

Substituting Eq. (4) into Eq. (2) and averaging all terms in the range $[t, t + 2\pi/\Omega]$, the equation for the slow variable is given as

$$\frac{dX}{dt} = \left[a - \frac{3bF^2}{2(\Omega^2 + a^2)}\right]X - bX^3 + f\cos(\omega t). \tag{5}$$

For the equivalent system in Eq. (5), there are two stable equilibriums $X^*_{\pm} = \pm\sqrt{[\frac{a}{b} - \frac{3F^2}{2(\Omega^2+a^2)}]}$ and one unstable equilibrium $X^* = 0$ for the case $a - \frac{3bF^2}{2(\Omega^2+a^2)} > 0$. There is only an unstable equilibrium $X^* = 0$ and the response will lose stability and diverge to infinity for the case $a - \frac{3bF^2}{2(\Omega^2+a^2)} \le 0$. By eliminating the constant component and retaining the harmonic terms in the response, we let $Y = X - X^*$ in which X^* is the stable equilibrium of Eq. (5). Then, substituting $Y = X - X^*$ into Eq. (5), we have

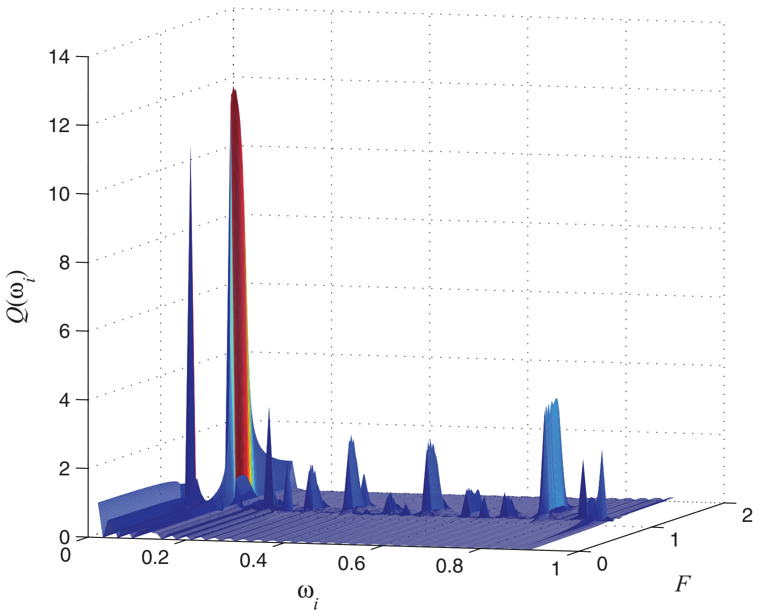

Fig. 1. The response amplitude $Q(\omega_i)$ versus the high-frequency excitation amplitude on the three-dimensional plane. The frequency ω_i is the frequency existing in the low-frequency band in the response. The simulation parameters are $a = 1.5$, $b = 1$, $f = 0.1$, $\omega = 0.12$ and $\Omega = 1$.

$$\frac{dY}{dt} = \left[a - \frac{3bF^2}{2(\Omega^2 + a^2)} - 3bX^{*2}\right]Y - 3bX^*Y^2 - bY^3 + f\cos(\omega t). \tag{6}$$

If we consider the superharmonic resonance in Eq. (6), then the term Y^2 may induce the frequency component $2m\omega$ and the term Y^3 may induce the frequency component $3n\omega$. Here, m and n are positive integer numbers. Besides these, some combined frequency components such as $p\omega + 2m\omega + 3n\omega$ may also exist, in which p is also a positive integer number. Through a different combination of p, m and n, we can get the superharmonic frequency components such as 2ω, 3ω, 4ω, 5ω, *etc*., in the response. This prediction has been verified in [12,13]. If we consider the subharmonic resonance in Eq. (6), then the term Y^2 may induce the frequency component $m\omega/2$ and the term Y^3 may induce the frequency component $n\omega/3$. Besides these, some other combined subharmonic frequency components such as $\omega + \omega m/2 + n\omega/3$ may exist. Based on this, we obtain some subharmonic frequencies in the response, for example, $\omega/3$, $\omega/2$, $2\omega/3$, *etc*. Furthermore, some other superharmonic frequencies such as $4\omega/3$, $3\omega/2$, $5\omega/2$, *etc*., may also exist. In the following analysis, we mainly focus on the resonance at the subharmonic and the superharmonic frequencies which have been never mentioned in the previous works done on VR.

To briefly verify the predictions above, we will take an example to show the existence of the subharmonic and superharmonic frequencies. In previous works reporting investigations on the VR phenomenon, the response amplitude is a quantitative index. It indicates the amplification of the weak low-frequency excitation when it passes through the nonlinear system. The response amplitude at the low-frequency of the excitation is defined by

$$Q(\omega) = \sqrt{Q_{\sin}^2(\omega) + Q_{\cos}^2(\omega)}/f, \tag{7}$$

where $Q_{\sin}(\omega) = \frac{2}{rT}\int_0^{rT} x(t)\sin(\omega t)dt$, $Q_{\cos}(\omega) = \frac{2}{rT}\int_0^{rT} x(t)\cos(\omega t)dt$, $T = 2\pi/\omega$ and r is a large enough integer number. Here, in order to describe the amplification effect on the subharmonic and superharmonic components, we rewrite Eq. (7) in a general form, *i.e.*,

$$Q(\omega_i) = \sqrt{Q_{\sin}^2(\omega_i) + Q_{\cos}^2(\omega_i)}/f, \tag{8}$$

where $Q_{\sin}(\omega_i) = \frac{2}{rT}\int_0^{rT} x(t)\sin(\omega_i t)dt$, $Q_{\cos}(\omega_i) = \frac{2}{rT}\int_0^{rT} x(t)\cos(\omega_i t)dt$, ω_i is the arbitrary frequency which is included in the response. In Fig. 1, the response amplitude $Q(\omega_i)$ versus the high-frequency excitation amplitude F is shown clearly. Importantly, Fig. 1 shows that the high-frequency excitation induces some subharmonic resonances and superharmonic resonances besides the primary resonance. It proves that the high-frequency excitation can induce the VsubR and VsupR. We will discuss this problem in a detailed manner in the next section.

In order to show the frequency components in the response in a much clear form, we provide Fig. 2 which displays the complexity of the response amplitude $Q(\omega_i)$ at the frequency ω_i. In Fig. 2 (a, c and d), the strongest resonance occurs at the primary frequency $\omega_i = \omega$. However, in Fig. 2 (b), the strongest resonance occurs at the subharmonic frequency $\omega_i = \omega/3$. This indicates that we may get an incorrect result if we only consider the primary frequency but ignore other frequencies in the response.

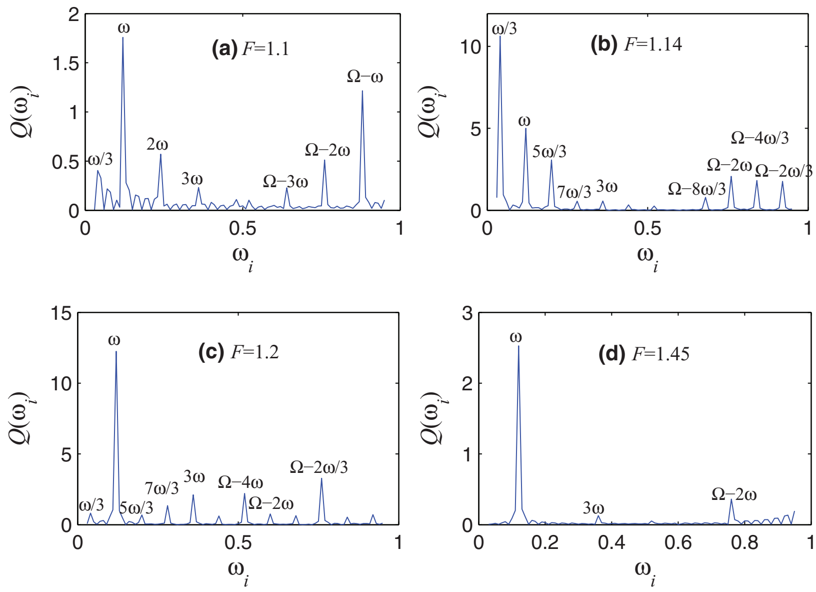

Fig. 2. The response amplitude $Q(\omega_i)$ versus the frequency ω_i for different values of *F*. The frequency ω_i is the frequency existing in the low-frequency band in the response. The simulation parameters are $a = 1.5,\ b = 1,\ f = 0.1,\ \omega = 0.12$ and $\Omega = 1$.

Also in Fig. 2, the subharmonic frequency $\omega/3$ and the superharmonic frequencies such as 2ω, 3ω, $5\omega/3$, $7\omega/3$ *etc.*, are clearly shown. For different values of *F*, the resonance frequencies and amplitudes may be very different. Besides the subharmonic and the superharmonic frequencies induced by the low-frequency and the nonlinear term in the system, there is another important fact in Fig. 2. Specifically, there are many other frequencies which are a combination of the high-frequency Ω and the integer or fractional multiple of the low-frequency ω. Some of them are labeled in Fig. 2. The high-frequency excitation also has effect on these combined frequencies. However, it is not the concerned topic in this paper. We cannot confuse the subharmonic or superharmonic frequency induced by the low-frequency ω with the combined frequencies generated by the combination of Ω and ω. For example, under the simulation parameters in Fig. 2, we have $\omega/3 = 0.04$ and $\Omega - 8\omega = 0.04$ too. However, we should not think that the subharmonic frequency at 0.04 is caused by $\Omega - 8\omega$. This is because the strength of the frequency component 8ω induced by the low-frequency excitation and the nonlinear term is extremely weak . This is especially proved by Fig. 2 (b), where the frequency component 8ω does not exist at all although the resonance at the subharmonic frequency $\omega/3$ is very strong. In general, the subharmonic frequency is mainly induced by the nonlinear term x^3. When the subharmonic frequency $\omega/3$ exists in the response, its cubic operation will bring the harmonic frequency ω to balance the excitation in the system. In this paper, we only consider the frequencies in the low-frequency band which are far away from Ω. We do not consider the frequency which is the combination of Ω and the integer or fractional multiple of ω.

3. Vibrational subharmonic resonance

At first, we recall the traditional VR phenomenon in Fig. 3 in which the VR induced by the parameter *F* and the linear term coefficient *a* is clearly shown. Similar figures are shown in many former references investigating the VR phenomenon. We show Fig. 3 here only in order to compare the VR at the primary frequency with that at the subharmonic and superharmonic frequencies. In the following of this section, we will study the VSubR at the frequency $\omega/3$ and $\omega/2$, respectively.

3.1. VSubR at the frequency $\omega/3$

The subharmonic component $\omega/3$ in the response is directly induced by the nonlinear term Y^3 in Eq. (6). When $\omega_i = \omega/3$, we get Fig. 4 via numerical simulation in which *a* and *F* are taken as control parameters. In Fig. 4, with the increase of *a* and *F*, the VSubR phenomenon occurs. However, comparing Fig. 4 with Fig. 3, the difference between the two figures is obvious. In Fig. 3, the three-dimensional surface of $Q(\omega)$ is continuous with a gradual increase of the control parameters. In Fig. 4, although the subharmonic resonance still occurs with the varying of the control parameters, the three-dimensional surface of $Q(\omega/3)$ appears to be discontinuous. The VSubR occurs at some discrete regions. Whether the VSubR occur or not depends on the value of *a*. Furthermore, with the increase of *F* and *a*, the peak value of $Q(\omega/3)$ will increase also. If we want to obtain a strong VSubR at the frequency $\omega/3$, we should choose a larger parameter *a* when *F* is considered as the single control parameter.

In order to show the amplification of the weak low-frequency excitation much more clearly, we show Fig. 5 in which the targets $Q(\omega)$ and $Q(\omega/3)$ are plotted as a function of the high-frequency excitation amplitude *F* and the linear term coefficient *a* respectively. In this figure, we find that the VSubR is very strong and it occurs at a single point, but not in a wide region.

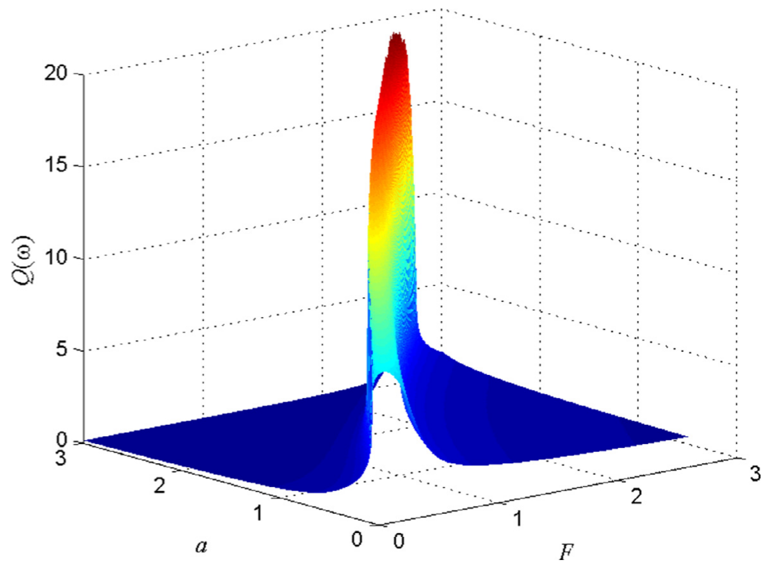

Fig. 3. The three-dimensional surface of the response amplitude $Q(\omega)$ versus the linear term coefficient a and the high-frequency excitation amplitude F, which indicates that the traditional VR occurs at the primary frequency ω. The simulation parameters are $b = 1,\ f = 0.1,\ \omega = 0.12$ and $\Omega = 1$.

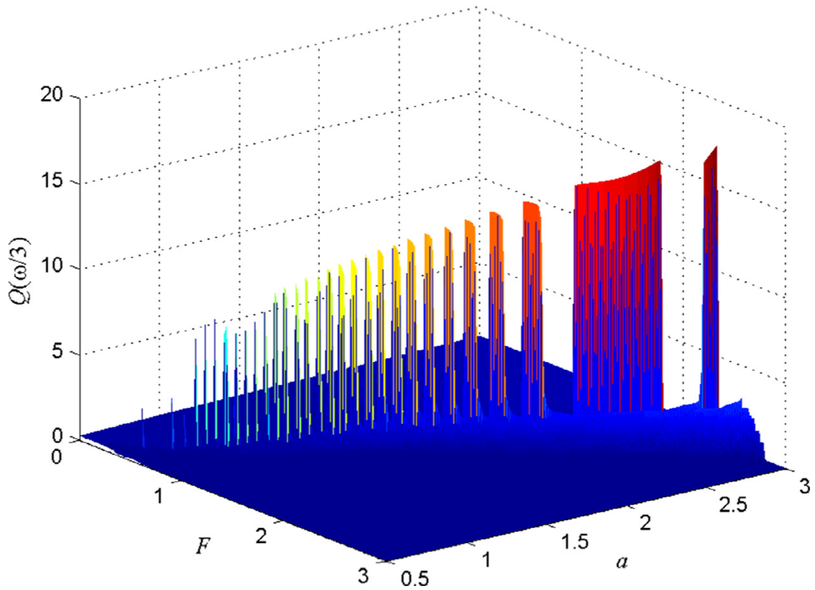

Fig. 4. The three-dimensional surface of the response amplitude $Q(\omega/3)$ versus the linear term coefficient a and the high-frequency excitation amplitude F, which indicates that the VSubR occurs at the subharmonic frequency $\omega/3$. The simulation parameters are $b = 1,\ f = 0.1,\ \omega = 0.12$ and $\Omega = 1$.

In Fig. 6, we fix the parameter a and consider F and b as the control parameters. The VSubR can be still induced by F or b in some discrete regions. With the increase of the nonlinear term coefficient b, the magnitude of the resonance peak does not increase. It is different from the effect of the linear term coefficient a on the magnitude of the resonance peak in Fig. 3.

In Fig. 7, the VSubR induced by the high-frequency excitation amplitude F or the nonlinear term coefficient b is shown in a two-dimensional plot, respectively. In Fig. 7 (a, b), the VSubR occurs at a smaller value of F for larger values of b. In Fig. 7 (c, d), the VSubR occurs at a smaller value of b for larger values of F. This fact can also be found in Fig. 6.

If the two harmonic excitations are fixed, the VSubR can also be induced by the linear term coefficient a and the nonlinear term coefficient b, as shown in Fig. 8. Similar to Fig. 4 and Fig. 6, the system parameter induced VSubR also occurs in some discrete regions or points.

As a conclusion of this subsection, we find that the strong VSubR at the frequency $\omega/3$ can be induced by the high-frequency excitation amplitude F, or the system coefficients a and b. This result has not been reported in previous references about VR.

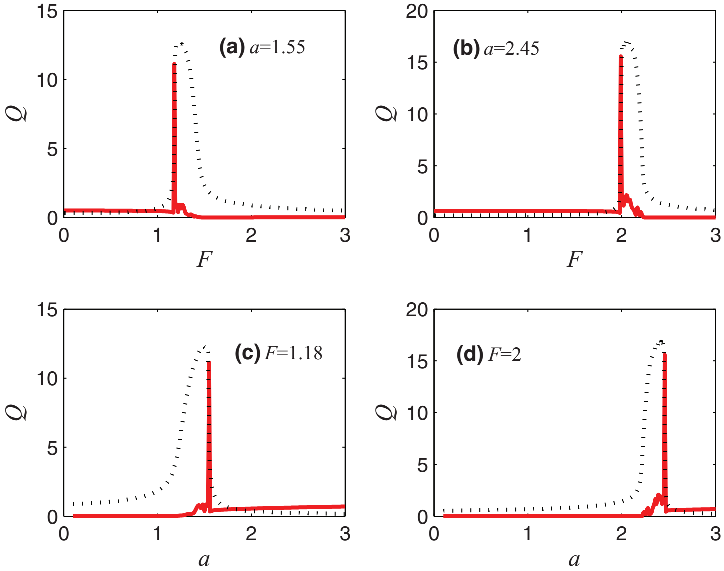

Fig. 5. Comparing VR and VSubR induced by the high-frequency excitation amplitude F in (a) and (b) and induced by the linear term coefficient$a$ in (c) and (d). The dotted line is $Q(\omega)$ and the continuous line is $Q(\omega/3)$ in each subplot. The simulation parameters are $b = 1,\ f = 0.1,\ \omega = 0.12,\ \Omega = 1$.

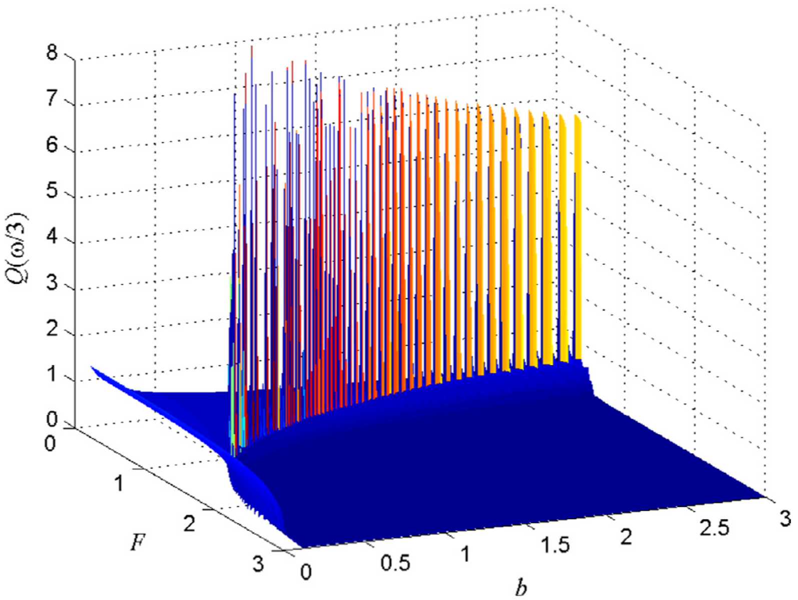

Fig. 6. The three-dimensional surface of the response amplitude $Q(\omega/3)$ versus the nonlinear term coefficient b and the high-frequency excitation amplitude F, which indicates that the VSubR occurs at the subharmonic frequency $\omega/3$. The simulation parameters are $a = 1,\ f = 0.1,\ \omega = 0.12$ and $\Omega = 1$.

3.2. VSubR at the frequency $\omega/2$

The subharmonic frequency $\omega/2$ in the response is induced by the quadratic term in the equivalent system. This term is included in Eq. (6). Considering a and F as the control parameters, the VSubR is shown in Fig. 9. In this figure, the VSubR also occurs at some discrete regions. However, comparing with the VSubR at the subharmonic frequency $\omega/3$ in Fig. 4, Fig. 6 and Fig. 8, the VSubR at the subharmonic frequency $\omega/2$ in Fig. 9 is weak. This is because the quadratic term is not included explicitly in the original system. Further, from Eq. (5), we know that the quadratic term does not exist if the high-frequency excitation is absent. This fact can also be found in Fig. 9. Nevertheless, under the high-frequency excitation, Fig. 9 shows the existence of the VSubR phenomenon induced by the high-frequency excitation amplitude F or the linear term coefficient a at the frequency $\omega/2$.

In Fig. 10, the VSubR at the subharmonic frequency $\omega/2$ is shown in a two-dimensional plot. For a fixed linear term coefficient a, the VSubR induced by the high-frequency excitation amplitude F is shown in Fig. 10 (a). At $F = 2.04$, the peak of the

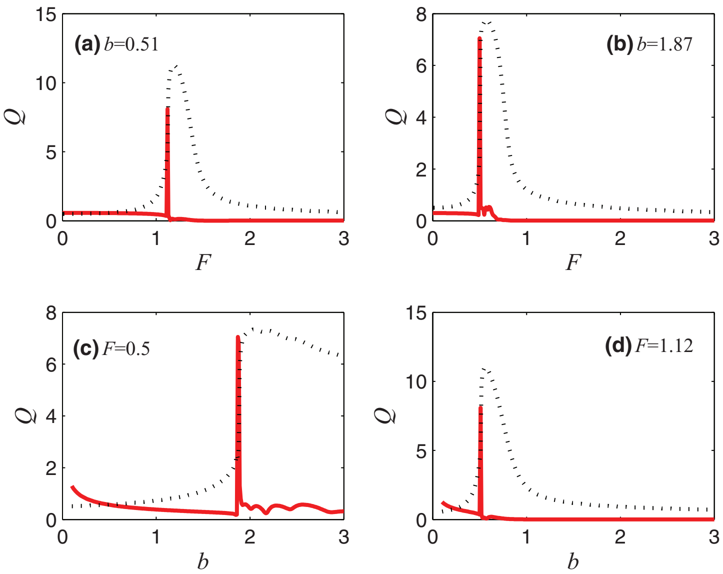

Fig. 7. Comparing VR and VSubR induced by the high-frequency excitation amplitude F in (a) and (b) and induced by the nonlinear term coefficient b in (c) and (d). The dotted line is $Q(\omega)$ and the continuous line is $Q(\omega/3)$ in each subplot. The simulation parameters are $a = 1$, $f = 0.1$, $\omega = 0.12$, $\Omega = 1$.

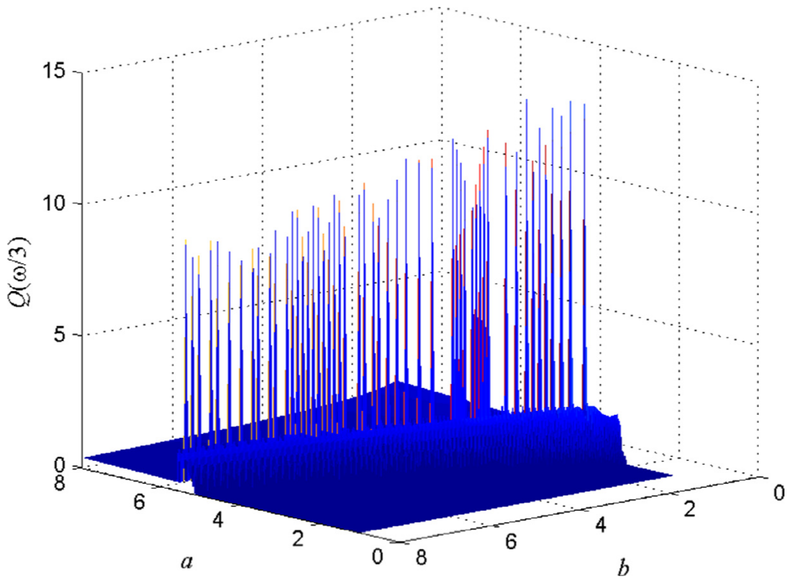

Fig. 8. The three-dimensional surface of the response amplitude $Q(\omega/3)$ versus the linear term coefficient a and the nonlinear term coefficient b, which indicates that the VSubR occurs at the subharmonic frequency $\omega/3$. The simulation parameters are $f = 0.1$, $F = 2$, $\omega = 0.12$ and $\Omega = 1$.

subharmonic resonance achieves the maximal magnitude. For a fixed high-frequency excitation amplitude F, the VSubR induced by the linear term coefficient a is shown in Fig. 10 (b). At $a = 2.56$, the subharmonic resonance occurs. In Fig. 10, it is also revealed that the VSubR appears at a single point but not in a wide region.

In Fig. 11, we consider the high-frequency excitation amplitude F and the nonlinear term coefficient b as the control parameters. The VSubR at the subharmonic frequency $\omega/2$ appears in this figure in some discrete regions. For small values of F and b, there might not exist the VSubR phenomenon. We make the weak VSubR to occur by choosing an appropriate combination of the two control parameters.

When the two excitations are fixed, the VSubR at the subharmonic frequency $\omega/2$ induced by the linear term coefficient a and the nonlinear term coefficient b is shown in Fig. 12. By increasing the linear coefficient term a, the critical value of the nonlinear term coefficient b which induces the subharmonic resonance, turns larger. In this subsection, we find that the VSubR at the subharmonic frequency $\omega/2$ can also be induced by the high-frequency excitation amplitude or the system parameters. The VSubR at the subharmonic frequency $\omega/2$ is much weaker than that at the subharmonic frequency $\omega/3$.

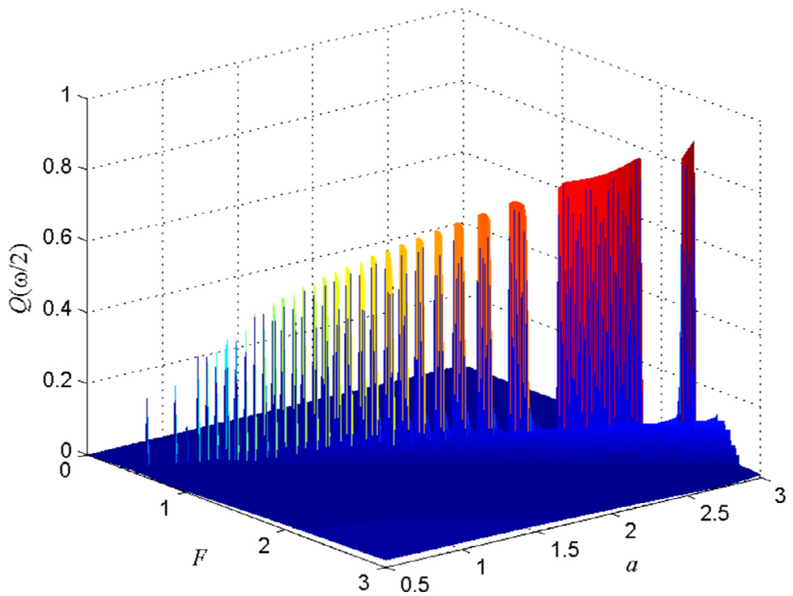

Fig. 9. The three-dimensional surface of the response amplitude $Q(\omega/2)$ versus the linear term coefficient a and the high-frequency excitation amplitude F, which indicates that the VSubR occurs at the subharmonic frequency $\omega/2$. The simulation parameters are $b = 1$, $f = 0.1$, $\omega = 0.12$ and $\Omega = 1$.

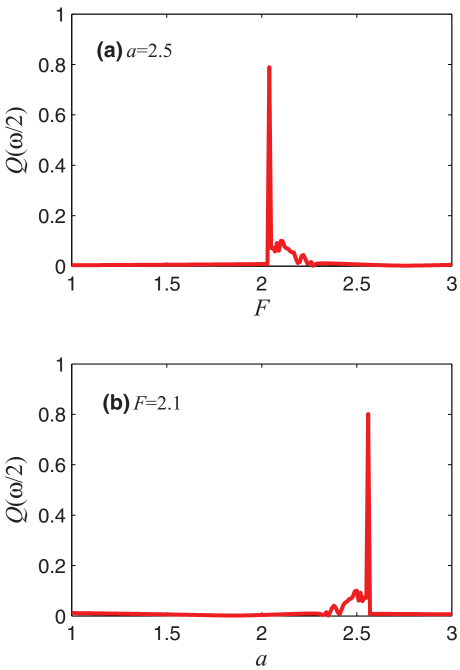

Fig. 10. The VSubR at the subharmonic frequency $\omega/2$ induced by the high-frequency excitation amplitude F in (a) and induced by the linear coefficient term a in (b). The simulation parameters are $b = 1$, $f = 0.1$, $\omega = 0.12$, $\Omega = 1$.

4. Vibrational superharmonic resonance

The VSupR at the superharmonic frequency $n\omega$ where n is a positive integer and $n > 1$ has been reported in [12] and [13]. However, based on the analysis in Section 2, we know that the VSupR can also occur at the superharmonic frequency $n\omega/m$, where $n/m > 1$, but is not an integer. In this section, we will verify this prediction. The VSupR at the superharmonic frequencies $4\omega/3$ and $5\omega/2$ will be shown respectively.

In Fig. 13, we show that the VSupR occurs at the superharmonic frequency $4\omega/3$. The VSupR can be induced by the high-frequency excitation amplitude F or by the linear term coefficient a. Although the VSupR is weak, this figure proves the existence

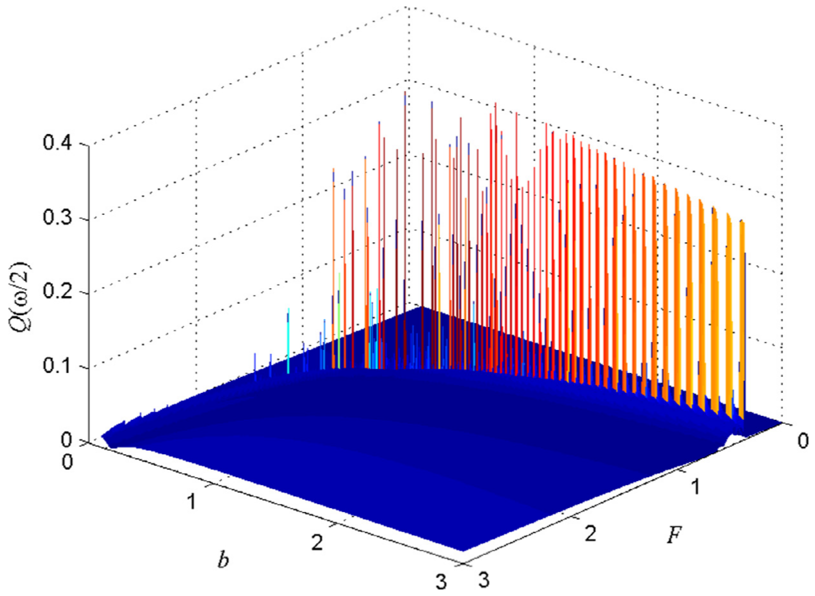

Fig. 11. The three-dimensional surface of the response amplitude $Q(\omega/2)$ versus the nonlinear term coefficient b and the high-frequency excitation amplitude F, which indicates that the VSubR occurs at the subharmonic frequency $\omega/2$. The simulation parameters are $a = 1,\ f = 0.1,\ \omega = 0.12$ and $\Omega = 1$.

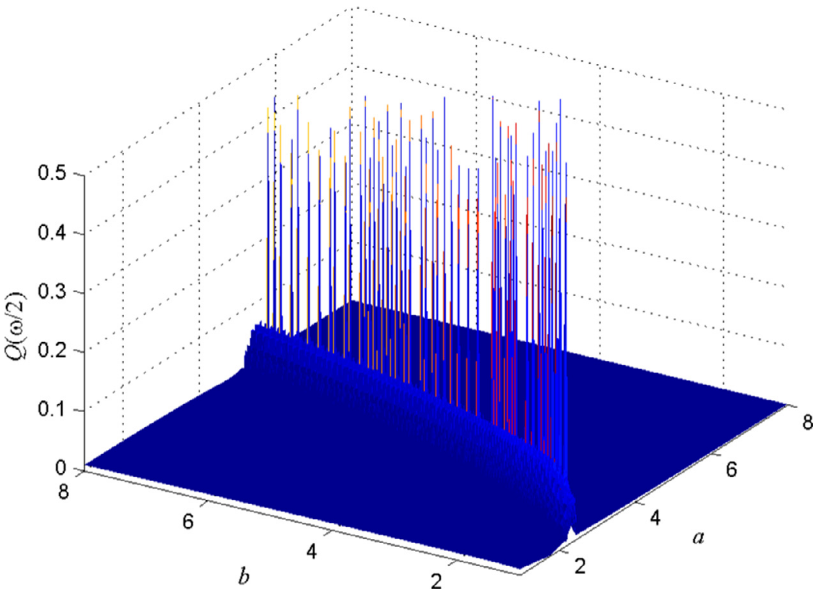

Fig. 12. The three-dimensional surface of the response amplitude $Q(\omega/2)$ versus the linear term coefficient a and the nonlinear term coefficient b, which indicates that the VSubR occurs at the subharmonic frequency $\omega/2$. The simulation parameters are $f = 0.1,\ F = 2,\ \omega = 0.12$ and $\Omega = 1$.

of the VSupR at the superharmonic frequency $4\omega/3$. The superharmonic frequency component $4\omega/3$ may be caused by the combination of the harmonic components ω and $\omega/3$.

In Fig. 14, the VSupR at the superharmonic frequency $5\omega/2$ induced by the high-frequency excitation amplitude F and the linear term coefficient a is clearly shown. The harmonic frequency component $5\omega/2$ may be induced by the harmonic frequency components combination $\omega/2$ and 2ω or by the combination of $\omega/2$ and 3ω. In this section, we find the VSupR, that in the same way as we commented earlier, it has not been reported previously on works related to VR.

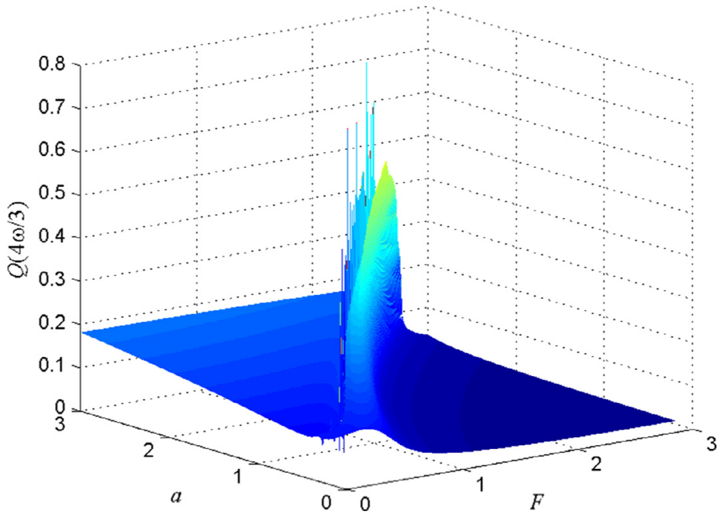

Fig. 13. The three-dimensional surface of the response amplitude $Q(4\omega/3)$ versus the linear term coefficient a and the high-frequency excitation amplitude F, which indicates that the VSupR occurs at the superharmonic frequency $4\omega/3$. The simulation parameters are $b = 1,\ f = 0.1,\ \omega = 0.12$ and $\Omega = 1$.

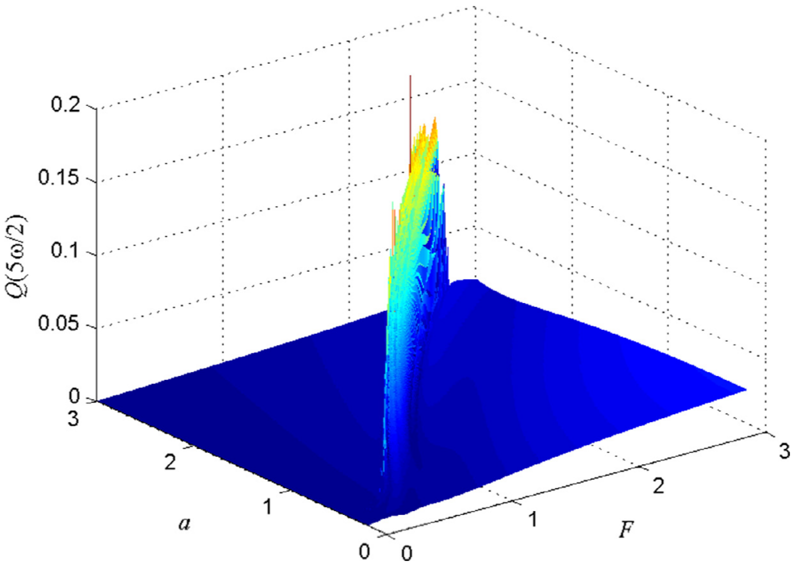

Fig. 14. The three-dimensional surface of the response amplitude $Q(5\omega/2)$ versus the linear term coefficient a and the high-frequency excitation amplitude F, which indicates that the VSupR occurs at the superharmonic frequency $5\omega/2$. The simulation parameters are $b = 1,\ f = 0.1,\ \omega = 0.12$ and $\Omega = 1$.

5. Discussions and conclusions

Ignoring the VSubR and the SupR phenomenon in a nonlinear system, may lead to incorrect results. For example, in an engineering structure, such as a beam under the combination of low-frequency and high-frequency excitations, if only the primary frequency in the response is considered, the structure may be safe at work. However, when the subharmonic and superharmonic frequency components and the primary frequency component are considered altogether, the response may induce a large deformation resulting in the damage of the structure, especially when the response amplitude at the subharmonic frequency is very strong as shown in Fig. 2 (b). In this paper, we have shown that the high-frequency excitation is a key factor to make the resonance to occur at the subharmonic frequency component. Certainly, it is a dangerous factor in the engineering structure. However, the effect of the high-frequency excitation on the response is usually ignored. This is because the high-frequency excitation induced response is very weak. Specifically, the response amplitude that can be obtained from Eq. (4) is very small. When the value of the high-frequency Ω in Eq. (4) is large, the response amplitude at Ω is small. In fact, the results in this paper show that the high-frequency excitation can induce VR, VSubR and VSupR in the response. Hence, the high-frequency excitation is also a dangerous factor to the engineering structure. Another example, in a neural system, when the primary resonance occurs in the system, it expresses a code. If the subharmonic resonance occurs, the firing pattern is different from the former. It indicates another code and transmits different information.

As a conclusion, some nonlinear vibrational resonance phenomena which we call *vibrational subharmonic resonance* (VSubR) and *vibrational superharmonic resonance* (VSupR) are reported in this work. The VSubR occurs at a subharmonic frequency, which is smaller than the low-frequency of the weak excitation. The VSupR occurs at a frequency which is larger than the low-frequency excitation. The VSupR can occur at a frequency which is an integer multiple of the low-frequency excitation, and it has been reported in previous references. In this paper, the VSupR mainly focuses on the superharmonic frequency which is larger but is not an integer multiple of the low-frequency excitation. The VSubR and VSupR can be induced by the high-frequency excitation or the system parameters. In the three-dimensional curves, the VSubR and VSupR usually appears in some discrete regions. Its nature is different from the traditional form. The VSubR and VSupR may be strong or weak. If the VSubR or VSupR occurs at the frequency component which is directly induced by a term in the original system, this kind of nonlinear VR may be strong. Otherwise, the nonlinear VR may be weak. Due to the importance of the nonlinear frequency response analysis in some scientific or engineering fields, the results shown in this paper might have a relevant value to better understand the complexity of the frequency components in the response of a nonlinear system subjected to multiple-frequency excitations.

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