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Parametric partial control of chaotic systems

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Abstract Discrete dynamical systems where one or several of their parameters vary randomly every iteration are usually referred to as random maps in the literature. However, very few methodologies have been proposed to control these kinds of systems when chaos is present. Here, we propose an extension of the partial control method, that we call parametric partial control, that can be naturally applied to random maps. We show that using this control method it is possible to avoid escapes from a region of the phase space with a transient chaotic behavior. The main advantage of this method is that it allows to control the system even if the corrections applied to the parameter are smaller than the disturbances affecting it. To illustrate how the method works, we have applied it to three paradigmatic models in nonlinear dynamics, the logistic map, the Hénon map and the Duffing oscillator.

Keywords Chaos control · Transient chaos · Random maps

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1 Introduction

A conventional way to think about chaos in engineering and control theory is to view it as an undesirable behavior that should be suppressed. For this reason, the main goal of controlling a system with a chaotic behavior has typically been to lock its dynamics into a periodic and predictable one [1,2]. But recently, there have appeared different scenarios where the maintenance of the chaotic behavior in systems with external disturbances cannot only be desirable but essential. In mechanics, for example, it is possible to avoid undesirable resonances with a chaotic dynamics [3]. In engineering, the thermal pulse combustor is more efficient in the chaotic regime [4]. In living organisms, chaotic dynamics is essential for some vital functions [5]. In biology, it has been suggested that the disappearance of chaos may be due in many cases to a pathological behavior [6]. However, there are systems where chaos is only a transient behavior, and the dynamics, after a chaotic motion in a certain region of the phase space, escapes to another state that could be highly undesirable. This is the case found in Ref. [7], where after a chaotic transient behavior, one of the species gets extinct, or in the cancer model described in Ref. [8], where the dynamics evolves toward an undesirable tumor growth.

With the motivation of sustaining the transient chaotic behavior indefinitely, different control methods have been suggested in the literature, which were mainly designed to be applied in deterministic systems. However, real systems are usually affected by some amount of noise, like unpredictable fluctuations of the system or external perturbations. Other sources of uncertainty can arise from the finite precision in the physical measurement of the state of the system and the finite precision in the applications of the control. In addition, the mathematical models are just an approach to the real dynamics and are normal to find some discrepancy. If we also consider that the computational simulations have a finite precision and therefore some deviations in the results, we can conclude that the presence of uncertainty is an unavoidable component in a control process.

One of the main features of a nonlinear chaotic dynamics is that little differences in the initial conditions of the system grow exponentially with time, and therefore, the presence of some uncertainty in these systems can play a critical role. To take advantage of this feature, a control method called *partial control* was proposed in Refs. [9,10]. This method includes a disturbance term in the scheme, which collects all the uncertainty present in the model. The main achievement of this method is that is able to keep the transient chaotic behavior forever, by using an amount of control smaller than the amount of disturbances affecting the system, which is a very counterintuitive result. The partial control method has been successfully applied to several paradigmatic systems like the Hénon map or the Duffing oscillator [9], as well as other models in the context of ecology or cancer dynamics [7,8].

In the classical partial control method, the disturbances and the control were applied directly on the phase space variables of the system, that is, $q_{n+1} = f(q_n, p) + \xi_n + u_n$. In this last equation, p represents the parameters of the system (which are supposed to be constant over time), ξ_n represents an additive disturbance term that varies randomly at each iteration of the map, and u_n represents the applied control. However, until now it has not been considered that the disturbances may affect some parameter p of the map. Here, we study a completely new control problem where the disturbances and the control terms are affecting directly some parameter of the system (instead of the phase space variables), that is, $q_{n+1} = f(q_n, p + \xi_n + u_n)$. For that reason, we call it *parametric partial control*. This study is motivated by the fact that the parameters usually fluctuate from one iteration to another in most real physical systems. These kinds of maps are called random maps in the literature. To study the fractal properties of these maps, the ideas of snapshot attractors [11] and pullback attractors [12] have been proposed. In the context of transient chaos, random maps are widely used to model systems where two different timescales dynamics coexist: one slow an predictable, and another with a small and fast fluctuating component. For example, this is the case in advective fluid dynamics [13], where the velocity field can be written as an average periodic field, plus a fluctuating component, or in some scattering processes [14–16] where the force field varies in time in a complex manner. As far as we know, the control scheme that we introduce here (parametric partial control) is the first that is able to sustain a transient chaotic dynamics in random maps.

The structure of the paper is as follows. In Sect. 2, we describe the partial control method for random maps, that is, we describe the parametric partial control. In Sect. 3, we apply the method to the logistic map, the Hénon map and the Duffing oscillator, as paradigmatic models in nonlinear dynamics. Finally, some conclusions are drawn.

2 The parametric partial control method

In this section, we present the parametric version of the partial control method that we call *parametric partial control*. This approach is based on the philosophy of the partial control method [10] with the difference that the disturbances are introduced in a parameter of the map instead of the variables. To apply it, first of all we have to identify the region Q_0 of the phase space where we want to keep the trajectories. The dynamics in this region will be

$$q_{n+1} = f(q_n, p + \xi_n + u_n), \tag{1}$$

where f is a function with a chaotic transient in Q_0, q is a point of Q_0, p is the central value of the parameter, ξ_n is a bounded disturbance $\xi_n \leq \xi_0$ and u_n is a bounded control, so that, $u_n \leq u_0 < \xi_0$. We say that $u_n \leq u_0$ is an *admissible control* and $\xi < \xi_0$ is an *admissible disturbance*. The trajectories satisfying these conditions with $u_0 < \xi_0$ will be called *admissible trajectories*. However, not all points of Q_0 can be controlled under these conditions, so it is necessary to implement an algorithm analogous to the *Sculpting Algorithm* [9], which removes the "bad" points. We say

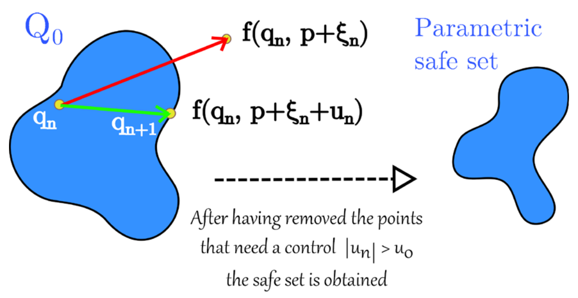

Fig. 1 Scheme of the parametric partial control. The *red arrow* shows the mapping of a point q, under the application of a random map in which a parameter p is affected by a bounded disturbance $|\xi_n| < \xi_0$. The *green arrow* shows the mapping of a point q, once the control u_n was applied to the parameter to keep the point in the blue region. Given the upper values of the disturbance ξ_0 and the control $u_0 < \xi_0$, the partial control method removes the points of the *blue region* that need a control $|u_n| > u_0$ for some possible $|\xi_n| < \xi_0$. For every point, we have to evaluate all possible disturbances $|\xi_n| < \xi_0$. Once the "bad" points are removed, a new region $Q_1 \subset Q_0$ is obtained. Iterating this process until it converges, we get a final region $Q_k \subset \cdots \subset Q_1 \subset Q_0$. We call this region, the *parametric safe set*

that a point q in Q is bad if for some admissible ξ, we have that $f(q, p + \xi + u)$ is not in Q. This allows us to define the parametric safe set operator as follows

$$\Gamma(Q) := \{\text{good points} \in Q\}. \tag{2}$$

This operator will remove the bad points of Q. But notice that not all the points that are good for Q will also be good for $\Gamma(Q)$. For this reason, we need to apply this operator recursively to find a set where all the points are good, that is,

$$Q_0 := Q; \quad Q_{n+1} := \Gamma(Q_n) \subset Q_n;$$
$$Q_\infty := \cap_{n=1}^{\infty} Q_n. \tag{3}$$

As a result, a set of good points Q_∞ is obtained. We call this set, the *parametric safe set*. In Fig. 1, we show graphically the procedure to check whether a point of the phase space is safe or not.

We have developed an algorithm to implement the parametric safe set operator on an arbitrary set Q of the phase space that has the following steps:

1. Select the region in phase space in which f has a chaotic transient. We notate the set of points of this region as the initial set Q_0. Then, we estimate the upper bound of the disturbance ξ_0, and we choose the upper bound of the control $u_0 < \xi_0$. Note that if the chosen u_0 is too small, the parametric safe set may be the empty set, and a bigger value of u_0 must be chosen.
2. For every point $q \in Q_i$ ($i = 0$ for the initial set), we need to check whether it is safe and can be part of an admissible trajectory or not. To do that, we compute $q_{n+1} = f(q_n, p + \xi_n + u_n)$ where the control u_n is applied with the knowledge of $p + \xi_n$, to place the trajectories back in Q_i, if it escapes, otherwise $u_n = 0$. For every point q_n, we have to check all possible disturbances ξ_n. If for all of them the absolute value of the applied control $|u_n|$ is smaller than u_0, then the point q is safe; otherwise, it is removed from Q_i.
3. After having removed all the points that do not satisfy the control condition, a new set $Q_{n+1} \subset Q_n$ is obtained. Then, we repeat again the step 2 with the new set Q_{n+1}. The process is repeated until it converges, in which case $Q_{n+1} = Q_n$, and this will be the *parametric safe set*.

Some practical considerations have to be done. In order to compute the parametric safe set, a finite grid covering Q_0 has to be used, since it is not possible to compute the infinite number of points in Q_0. For an analogous reason, only a finite sample of disturbances ξ_n can be checked for every point q. We will call the grid resolution as the distance between two adjacent points q, and the parameter resolution as the distance between two adjacent values of the parameter affected by different disturbances. Higher resolutions give a more accurate parametric safe set. In this sense, we have found that beyond a critical resolution of the grid of Q and ξ, the safe set remains unchanged. For that reason and from a practical point of view, we recommend to compute the safe set with the algorithm proposed with increasing resolutions until finding the critical value for which the shape of the safe set found remains unchanged. That one will be a very good approximation of the real safe set.

The parametric safe set obtained using the algorithm just described is a positively invariant set [17]. That is, if the controlled system's state is at some time inside the parametric safe set, then it will also be contained again in this set in the future. However, the system is not invariant since the same property does not apply when going backwards in time.

3 Application of the parametric partial control method

In order to show how the method works, we have considered three paradigmatic models in nonlinear dynamics, the 1D logistic map, the 2D Hénon map and the Duffing oscillator, all of them for a choice of parameters where transient chaos is present. In all cases, we consider that the parameter is affected by a disturbance with a uniform probability distribution $|\xi_n| \leq \xi_0$. But any other distribution is possible, provided that it is bounded. The condition to apply the control is $|u_n| \leq u_0$, and therefore, many choices are possible. In the following, we have taken the minimum allowed control to stay in the parametric safe set, but other criteria are possible.

3.1 The logistic map

The logistic map is a very well known 1D map and is defined as follows:

$$x_{n+1} = r x_n (1 - x_n). \quad (4)$$

For a parameter value $r \in [0, 4]$, the interval $x \in [0, 1]$ maps to itself. However for $r > 4$, the orbits starting in this interval escape toward infinity after a chaotic motion (see Fig. 2). With the aim of keeping the trajectories in $Q_0 = [0, 1]$ and assuming that the parameter

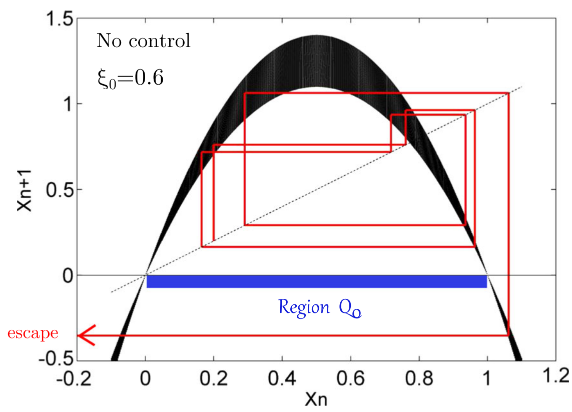

Fig. 2 Uncontrolled orbit in the logistic random map. The parameter $r = 5$ is affected by disturbances with upper bound $\xi_0 = 0.6$. The *black wide curve* is obtained for all possible values of the parameter, $r \in [5 - \xi_0, 5 + \xi_0]$, of the logistic map. In *red*, we show an example of an uncontrolled trajectory that after a chaotic motion in Q_0 escapes to minus infinity

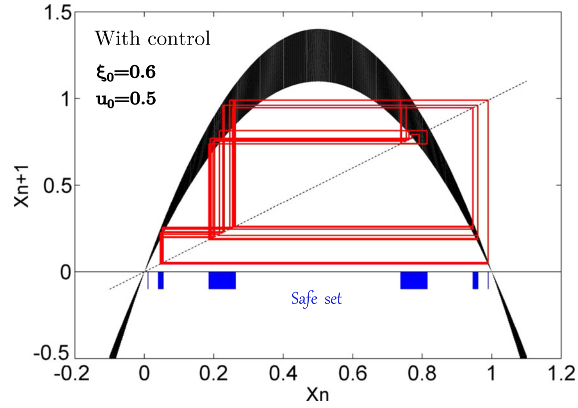

Fig. 3 Partially controlled orbit for the logistic random map. We apply the partial control method to the logistic map, with $\xi_0 = 0.6$ and $u_0 = 0.5$ and a grid resolution of 0.001, to obtain the parametric safe set which is shown with the *wide blue* segments to help the visualization. The orbits starting in this set remain there after applying a control $u_n \leq 0.5$ every iteration. In *red*, we show an example of a partially controlled trajectory. We are plotting only 50 iterations

is affected by some disturbances $|\xi_n| \leq \xi_0$, the parametric partially controlled dynamics for this map can be written as

$$x_{n+1} = (r + \xi_n + u_n) x_n (1 - x_n), \quad (5)$$

where $|u_n| \leq u_0 < \xi_0$ is the control applied. To show an example of how the method works, we have taken the values $r = 5$, $\xi_0 = 0.6$ and $u_0 = 0.5$. After the computation of the algorithm described in the previous section, we have obtained the parametric safe set shown in Fig. 3. The blue wide segments represent the safe points of x. In this figure, it has also been displayed a partially controlled trajectory (in red), which as can be seen remains chaotic and within Q_0 indefinitely.

The appearance of the safe sets follows a very characteristic pattern when the disturbance constrain is varied. In Fig. 4, we explore this feature computing different safe sets for different disturbances ξ_0 and putting all together. The upper parametric safe set corresponds to a large value of the disturbance, and we gradually reduce it. The control value u_0 selected in each case is close to the minimum value for which the parametric safe set exists (smaller values give the empty set). The Cantor set structure of the different safe sets is related to the fractal structure of the chaotic saddle of the map. This feature reveals that the control problem is approximately self-similar in the sense that each value

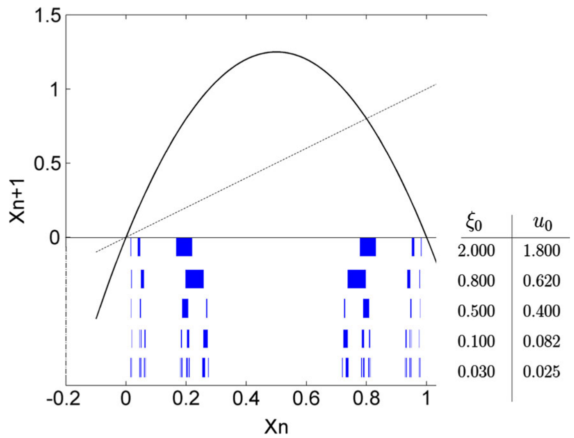

Fig. 4 Different safe sets computed for different amounts of the disturbance constraint ξ_0 affecting the parameter r. Each safe set was computed with the corresponding value of disturbance and control shown on the *left side* of the picture. The grid and parameter resolution used is 0.001. We can see the Cantor structure of the parametric safe set, as the disturbance decreases. In all cases, the upper control bound selected u_0 is close to the smallest possible. Under these values, no parametric safe set exists

of disturbance is linked with a convenient scale where the parametric safe set is located. Due to this property, the control method is robust and efficient.

3.2 The Hénon map

The Hénon map is a 2D map defined by

$$x_{n+1} = a - by_n - x_n^2$$
$$y_{n+1} = x_n. \tag{6}$$

This map shows transient chaos for a wide range of the parameters a and b. We have chosen here the parameter values $a = 2.16$ and $b = 0.3$. For these parameters, the trajectories with initial conditions in the square $[-4, 4] \times [-4, 4]$ have a very short chaotic transient, before finally escaping of this region toward infinity. An example of this behavior is shown in Fig. 5 for a given initial condition. We consider now, a situation where the parameter b is affected by some disturbance $|\xi_n| \le \xi_0$. To keep the orbits in $Q_0 = [-4, 4]\times[-4, 4]$, we apply a control $|u_n| \le u_0 < \xi_0$, so that the controlled dynamics can be described as:

$$x_{n+1} = a - (b + \xi_n + u_n)y_n - x_n^2$$
$$y_{n+1} = x_n. \tag{7}$$

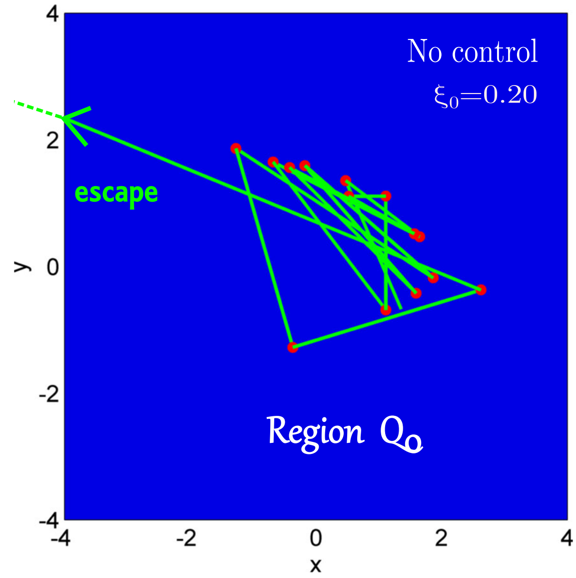

Fig. 5 An uncontrolled trajectory in the Hénon random map with $a = 2.16$ and $b = 0.3$. The parameter b is affected by disturbances with upper bound $\xi_0 = 0.20$. The *blue square* $[-4, 4] \times [-4, 4]$ is the region Q_0. In the absence of an external control, the trajectories in Q_0 escape outside the square after a very short chaotic transient. An example of an uncontrolled trajectory is displayed with the *red points* connected by the *green lines* to help to see the evolution

In order to show how the method works in the Hénon map, we have taken $\xi_0 = 0.20$ and $u_0 = 0.15$ as the disturbance and control constrains, respectively. Next, we have applied the algorithm to find the parametric safe set with these values. After 8 iterations of the algorithm, the process converges and the parametric safe set is obtained. In Fig. 6, the blue points are the safe points, while the blank points are the points that have been removed, since they do not satisfy the control conditions. It is also shown a partially controlled orbit (red points), which remains chaotic in the square forever.

In order to show how the parametric safe set changes for a different value of ξ_0, we display in Fig. 7 the parametric safe set obtained when $\xi_0 = 0.050$ and $u_0 = 0.036$. Again a partially controlled trajectory is shown in red. This system is affected by smaller disturbances, and as a consequence, the parametric safe set is more complex. The tendency as the disturbance decreases is that the parametric safe set becomes more and more complex due to the fractal structure of the chaotic saddle underlaying the dynamics. For this reason, more and more resolution is necessary to solve these kinds of safe sets. However, we always have a

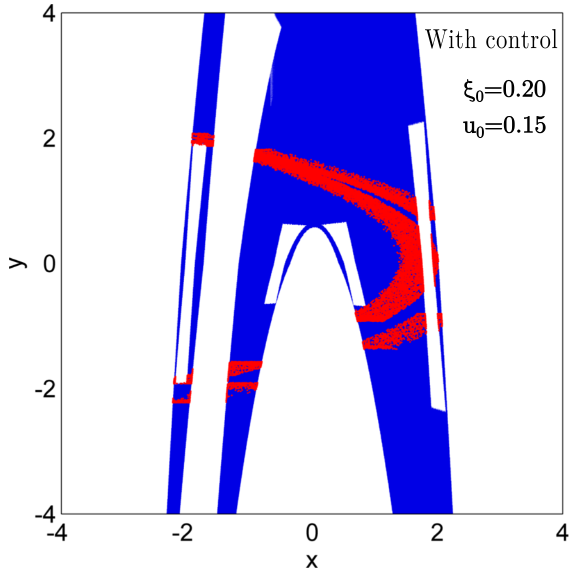

Fig. 6 Controlled trajectory in the Hénon random map with $\xi_0 = 0.20$. Same situation as shown in Fig. 5. The partial control method has been applied to keep trajectories in Q_0 forever. The upper bound of control is $u_0 = 0.15$. The grid resolution taken is 0.01, and the parameter resolution is 0.005. As a result, the parametric safe set (in *blue*) is obtained. All the orbits of the map starting in the blue set remain there after the application of controls smaller than $u_0 = 0.15$. The *red points* display a partially controlled trajectory, where 20,000 iterations of the trajectory have been plotted

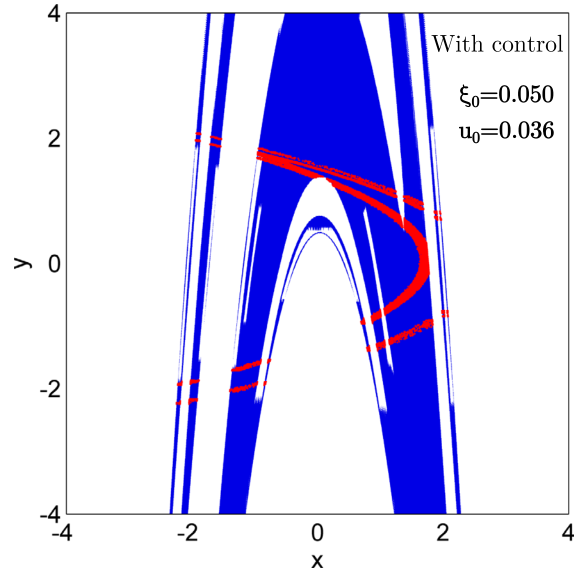

Fig. 7 Controlled trajectory in the Hénon random map with $\xi_0 = 0.050$. For this case, the upper value of control is $u_0 = 0.036$, the grid resolution used is 0.001, and the parameter resolution is 0.0005. In we compare it with the previous figure, we see that the appearance of the parametric safe set is more complex, due to fact that the disturbance value is smaller. It has been also plotted (in *red*) 20,000 iterations of a partially controlled trajectory

finite resolution in the computation, so the value of the disturbance can never be zero.

3.3 The Duffing oscillator

The partial control method can also be applied to maps built from continuous dynamical systems. We have considered, as an example, the Duffing oscillator for a choice of parameters in which transient chaotic trajectories are present.

$$\ddot{x} + 0.15\dot{x} - x + x^3 = 0.245sin(t). \tag{8}$$

Due to the periodic forcing, it is possible to built a stroboscopic map, the time-2π map, where the flow is cut every $\Delta t = 2\pi$. The transient chaotic dynamics is captured in the square $[-2, 2]\times[-2, 2]$ shown in Fig. 8. Without external control, almost all initial conditions in this region, after a chaotic behavior, fall in one of the three attractors present in the phase space. The system has two period-1 attractors and one period-3 attractor, as shown in the figure.

With the aim of keeping the trajectories far from these attractors, we have applied the partial control method further considering that the forcing amplitude is affected by some bounded disturbance $|\xi_n| \leq \xi_0$. Applying the control $|u_n| \leq u_0$ in the same parameter as well, the amplitude of the forcing varies according to $0.245 + \xi_n + u_n$ every iteration.

As an example, we have computed the safe set for the upper bound value of the disturbance $\xi_0 = 0.020$ and the upper bound of control $u_0 = 0.014$. We have used a grid of 1000×1000 in the square $[-2, 2] \times [-2, 2]$, where the balls centered in each attractor has been removed to prevent the periodic behavior. Taking this initial region as Q_0, the Sculpting Algorithm was applied. The safe set obtained is shown in Fig. 9, where a controlled trajectory (30,000 iterations in red) also appears. Notice that the partially controlled trajectory is chaotic and never fall in the attractors.

3.4 More dimensions, more parameters

We have computed safe sets for the logistic map, the Henón map and the Duffing oscillator, but no theoreti-

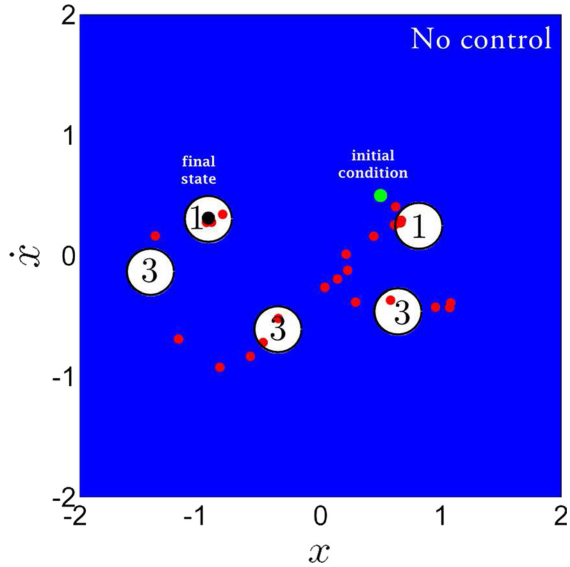

Fig. 8 Escapes in the Duffing oscillator. The *blue square* $[-2, 2] \times [-2, 2]$ represents a stroboscopic section of the Duffing oscillator for a choice of parameters where transient chaos is present. In this situation, almost all trajectories eventually fall in some of the attractors (*white holes*) showed in the figure. For this case, there are two period-1 attractors and one period-3 attractor. The *red dots* in the figure represent an uncontrolled trajectory where the forcing amplitude is affected by the disturbance $\xi_0 = 0.020$. After a few iterations, the trajectory eventually reaches one of the attractors

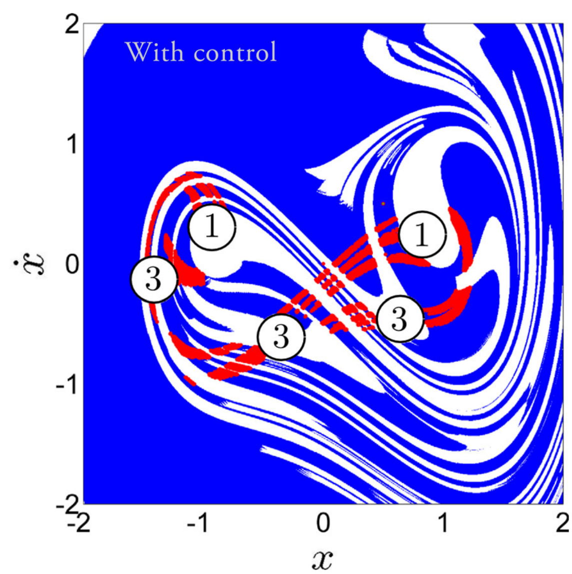

Fig. 9 Controlled trajectory in the Duffing oscillator with $\xi_0 = 0.020$ and $u_0 = 0.014$. After removing the holes, corresponding to the attractors, the safe set (in *blue*) was computed with a grid of 1000×1000, (grid resolution 0.004, parameter resolution 0.0002). The *red dots* represent a controlled trajectory made up of 30,000 iterations in the Stroboscopic map

cal restrictions exist for the application of the parametric partial control method in other maps with higher dimensions. The principal drawback is the extra time of computation, since the points that have to be analyzed grow exponentially with the dimension.

On the other hand, situations where more than one parameter is affected by random disturbances are possible. The scheme of the method is easily expandable; for example, in the case of m parameters $p^1, p^2, \ldots, p^m$, the partially controlled dynamics would be described as

$$q_{n+1} = f\left(q_n, \left(p^1 + \xi_n^1 + u_n^1\right), \left(p^2 + \xi_n^2 + u_n^2\right), \ldots, \left(p^m + \xi_n^m + u_n^m\right)\right), \quad (9)$$

with the conditions

$$\sqrt{\left(\xi_n^1\right)^2 + \left(\xi_n^2\right)^2 + \cdots + \left(\xi_n^m\right)^2} \le \xi_0$$
$$\text{and} \quad \sqrt{\left(u_n^1\right)^2 + \left(u_n^2\right)^2 + \cdots + \left(u_n^m\right)^2} \le u_0 < \xi_0. \quad (10)$$

Again, the extra parameters increase considerably the computational time to obtain a parametric safe set. One way to accelerate the computations would be to parallelize the algorithm, but is not trivial because a point is safe or not depending on the rest of the safe points in Q_0. Another possibility would be to built an adaptive grid which increases the resolution in the parts where the parametric safe set has a complex boundary. There is another way to reduce the computational time, consisting in finding the parametric safe set only in the region where the system had a long transient dynamics. To do that, a big amount of initial conditions would have to be taken, and after removing the first iterations, we would have only the points that stay several iterations in the region of interest. Taking a grid that covers this region, it is possible to economize the computation, with the counterpart of losing information about the final parametric safe set.

Once the parametric safe set is obtained, the computation of the partially controlled trajectory is very fast, in part because the control condition $|u_n| \le u_0$ is not too restrictive. In the previous examples, we have taken the minimum control criterion, but in fact, we can explore randomly the points q_{n+1} of the parametric safe set which satisfy this condition and select the first one that we find. This strategy also reduces the computational time while increases the entropy of the partially controlled trajectory, which can be a desirable feature in problems that pursue to increment the diffusion or mixing rates.

4 Conclusions

In this work, we have presented a new control method that we call parametric partial control of chaotic systems, which is an extension of the partial control method. This method is designed to be applied on systems that exhibit a chaotic transient behavior in certain region Q_0 and affected by disturbances. Taking advantage of the structure of the chaotic saddle present in the phase space, it is possible to find certain subsets of Q_0 which are advantageous in a control sense. Roughly speaking, trajectories starting in this set need a smaller control to remain in this set (even smaller than the disturbances). We call this set, the parametric safe set.

The parametric partial control is applied on maps where some parameter is affected by a bounded disturbance ξ_0. The goal of the method is to keep the transient chaotic motion forever in Q_0, by the application of a bounded control u_0 in the parameter. With the application of the algorithm described, we show that it is possible to obtain safe sets, for values $u_0 < \xi_0$, which is the most relevant result.

We have successfully applied the parametric partial control to the 1D logistic random map, the 2D Hénon random map and the Duffing oscillator with some disturbance affecting the forcing amplitude. In all the systems considered, we have taken a choice of parameters where transient chaos is present. We have computed the parametric safe sets for different values of the disturbance, showing how the parametric safe set changes with it. This feature gives the method a high versatility and robustness under changing circumstances. Finally, we have analyzed the hypothetical problems of extra dimensions and extra parameters in the maps. No theoretical restrictions exist for these cases, and the only limitation is the computational resources.

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