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Adaptive piecewise re-scaled stochastic resonance excited by the LFM signal

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Abstract The piecewise re-scaled stochastic resonance method is proposed and thoroughly investigated in a bistable system, which is induced by the linear frequency-modulated (LFM) signal. At first, the theoretical formulation for piecewise re-scaled stochastic resonance is explained in detail. Then, several numerical simulations are carried out and the effects of some related parameters are discussed, in which the moment of the signal segmentation and the re-scaled coefficient are key factors. Meanwhile, the numerical results indicate that the proposed method manages to process the LFM signal submerged in the noise. After that, adaptive piecewise re-scaled SR is proposed to solve the problem of the parameter selection. At last, the comparison between fractional Fourier transform (FRFT) and the proposed method is present. Compared to the traditional FRFT, the method has a better performance, especially in amplification effect. The method in this paper may provide reference for processing other kinds of frequency-modulated signals besides the LFM signal.

1 Introduction

Stochastic resonance (SR) has attracted a lot of attention in different research fields for many years. A weak signal can be improved excellently with the cooperation between the nonlinear term and a certain amount of noise by SR [1]. In the traditional SR theory, small parameter SR needs to meet the preconditions, i.e., the orders of magnitude of the system parameters are 1, and the weak signal is a low-frequency harmonic signal. If these preconditions cannot be satisfied, SR can be induced by some special techniques. Among them, the re-scaled method is an effective way. In the re-scaled SR theory, choosing the proper system parameters makes

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the system match the signal frequency and then leads to SR. The normalized scale transform [2] and the improved general transform [3] are common re-scaled methods. They have also been used successfully in the fault diagnosis field of a rotary machine [4–8].

However, besides the harmonic signal, there are many kinds of aperiodic signals needing to be processed, such as the aperiodic binary signal, the linear frequency-modulated (LFM) signal, the nonlinear frequency-modulated (NLFM) signal, etc. In general, signals are usually submerged in the noise. Hence, it is important to study SR for these kinds of signals. There are some investigations for aperiodic SR in the system subjected to binary signals [9–11]. Due to the significance of the modulated signals, especially the LFM signal widely used in the scientific and engineering fields [12–17], it is necessary to study SR in the system excited by the LFM signal. As is known to all, the frequency of the LFM signal increases with the time. It results in the problems that the re-scaled method cannot induce SR. At present, FRFT is a tool for separating the LFM signal from the noise [18,19]. However, through FRFT, the LFM signal submerged in the noise may be destroyed or weakened to some extent, during the de-noising process. Moreover, even if the pure LFM signal can be separated well, it only separates the weak LFM signal but do not recover and enhance the signal well. Previously, we proposed a vibrational method to amplify a LFM signal. However, the noise was not considered in that work [20]. To our knowledge, noise almost exists in all the situations. As a result, it is urgent to develop a new technology that not only can remove the noise but also can enhance the LFM signal. Motivated by this, we will give an improved technique to induce SR by the traditional re-scaled method, in order to enhance the LFM signal. In addition, to improve the efficiency of the proposed method, adaptive particle swarm optimization (APSO) algorithm is used with good stability and quick convergence.

The rest of the paper is organized as follows. In Sect. 2, the theory for piecewise re-scaled SR is presented and some numerical simulations are investigated. In Sect. 3, adaptive piecewise re-scaled SR is proposed by using APSO algorithm. In Sect. 4, the comparison between FRFT algorithm and the proposed method is presented. In Sect. 5, the main results of this work are concluded.

2 Piecewise re-scaled SR method

The classic bistable system for SR is governed by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax - bx^3 + s(t) + \xi(t), \tag{1}$$

where system parameters $a > 0$, $b > 0$. $\xi(t)$ is the Gaussian white noise, which is described as

$$\langle \xi(t) \rangle = 0, \ \langle \xi(t_1)\xi(t_2) \rangle = \sigma^2 \delta(t_1 - t_2) \tag{2}$$

where σ is the noise intensity. $s(t)$ is the LFM signal given by

$$s(t) = A\cos(\pi\gamma t^2 + 2\pi f t + \phi), \tag{3}$$

where A, γ, f and ϕ are the amplitude, the chirp rate, the centroid frequency and the initial phase of the LFM signal, respectively. Here is the time series of $s(t)$, as shown in Fig. 1.

With the increase in the time, the instantaneous frequency of the LFM signal turns large. So there are more and more high-frequency components appearing in the signal as the time goes. According to small parameter SR, it cannot satisfy the precondition that SR is able to occur. To solve such problems, system parameters must match signal parameter well so that SR can

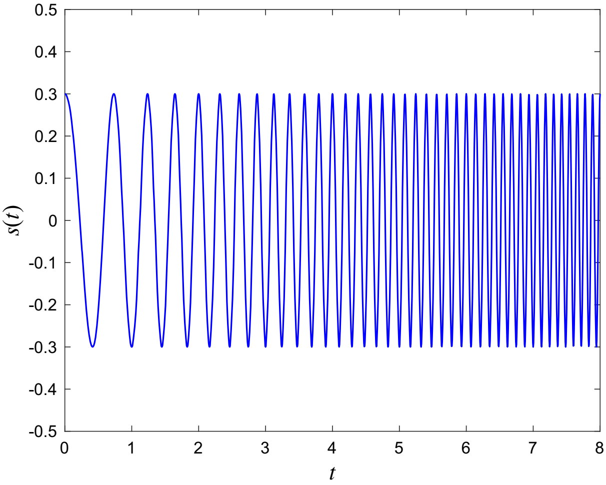

Fig. 1 Time series of $s(t)$, $A = 0.3$, $\gamma = 1$, $f = 1$ and $\phi = 0$

occur easily and we thus introduce the piecewise and rescaling idea. First of all, we divide the whole signal into several segmentations and choose different scale parameters for each segmentation. Then, we input these segmentations into the system to obtain several groups of new time series. Finally, we combine these time series to get the eventual response. Note that, throughout the paper, the input signal is always divided into 5 segmentations equally in the time domain, that is, each segmentation has the same length.

For the LFM signal in Eq. (3), its instantaneous frequency f_t is

$$f_t = \gamma t + f. \tag{4}$$

We define a scale parameter β in the considered signal segmentation,

$$\beta = \beta_0(\gamma t_{ei} + f), \tag{5}$$

where β_0 is the re-scaled coefficient, and t_{ei} labels the moment of the ith signal segmentation. Here, the general scale transformation is selected as a re-scale method. As a result, the system by the re-scaled method is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \beta a x - \beta b x^3 + \beta s(t) + \beta \xi(t). \tag{6}$$

Note that a and b are small, while the system parameters βa and βb are large. SR by Eq. (6) is often called large parameter SR [21], which can process the high-frequency signal. Then SR for the LFM signal can be achieved.

We use the cross-correlation coefficient C_{sx} to quantify SR [22–24]. Specifically, C_{sx} is calculated by

$$C_{sx} = \frac{\sum_{j=1}^{n}[s(j) - \bar{s}][x(j) - \bar{x}]}{\sqrt{\sum_{j=1}^{n}[s(j) - \bar{s}]^2 \sum_{j=1}^{n}[x(j) - \bar{x}]^2}}, \tag{7}$$

where $\bar{s}$ and $\bar{x}$ are the average of the input LFM signal and the system output, respectively. The specific procedure on how to measure SR is as follows. First, we adopt an easy way to construct the output $x(t)$. Specifically, we connect all output segmentations together according to their sequence to obtain the whole output time series $x(t)$. Then, $x(t)$ is substituted into Eq. (7) to

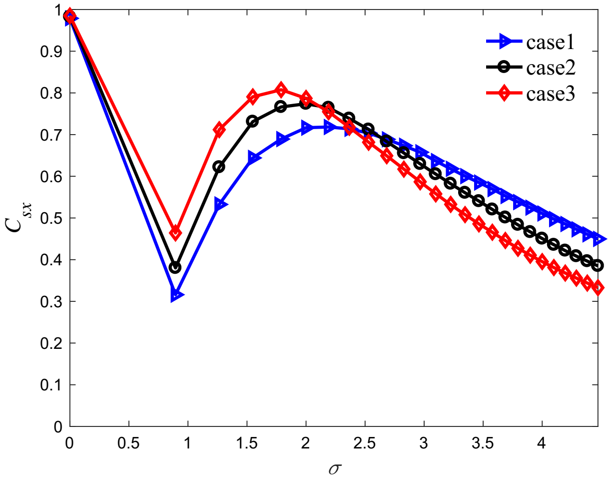

Fig. 2 The cross-correlation coefficient C_{sx} versus the noise intensity σ. Case 1, case 2, case 3 corresponding to t_{ei} are the initial point, the middle point, the end point of each segmentation, respectively. Other simulation parameters are $A = 0.3$, $\gamma = 1$, $f = 1$, $\phi = 0$, $a = 1$, $b = 1$ and $\beta_0 = 600$

calculate C_{sx}. SR becomes optimal when C_{sx} reaches a peak value. In the following analysis, the fourth Runge–Kutta algorithm is applied to calculate the stochastic differential equation numerically. In Monte Carlo simulations, we usually test at least hundreds of trials. Therefore, we use 100 different noise realizations to calculate C_{sx} and use their mean value as the final cross-correlation coefficient.

To present the method proposed in the last section, we investigate the influence of different choices of t_{ei} on the value of the cross-correlation coefficient. In Fig. 2, we choose three different cases. Case 1, case 2, case 3 corresponding to t_{ei} are the initial point, the middle point, the end point of each segmentation, respectively. There are several evident facts shown in Fig. 2. Apparently, SR is able to happen, which means SR in the LFM signal excited system is achieved. In other words, the LFM signal can be improved by the cooperation between the noise and the nonlinear system, based on the re-scaled theory. Furthermore, we find that case 3, i.e., t_{ei} corresponding to the end point of each segmentation, can make the cross-correlation to reach a larger value of the maximal peak. It implies optimal SR output. As a result, we choose the endpoint of each segmentation as t_{ei} in the rest of the paper.

In Fig. 3, under different signal amplitudes, the phenomenon of SR is studied. In this figure, for a stronger signal excitation, we can get a larger peak value of the cross-correlation coefficient.

In Fig. 4, the effect of the re-scaled coefficient β_0 on the cross-correlation coefficient C_{sx} is studied. The curves show that with the increase in β_0, C_{sx} may have a larger peak value. It is because for larger value of β_0, the instantaneous frequency is lower at the same time point t.

To illustrate SR excited by the LFM signal further, in Fig. 5, we give the time series corresponding to the maximal peak value of Fig. 4. On the one hand, the input/output synchronization is clearly shown and the output is much stronger than the pure LFM signal. It demonstrates the occurrence of SR. On the other hand, with the evolution of the output, we find that the amplitude of the output decreases gradually. It is because the coefficient is computed by comparing the whole output with the whole input signal. Through the calculation,

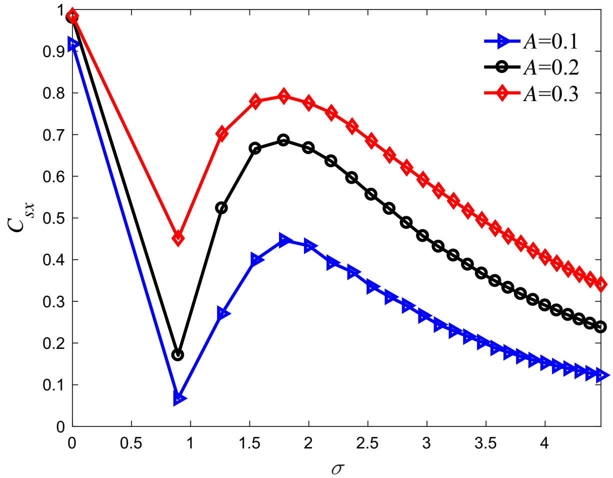

Fig. 3 The cross-correlation coefficient C_{sx} versus the noise intensity σ. Other simulation parameters are $\gamma = 1$, $f = 1$, $\phi = 0$, $a = 1$, $b = 1$ and $\beta_0 = 600$

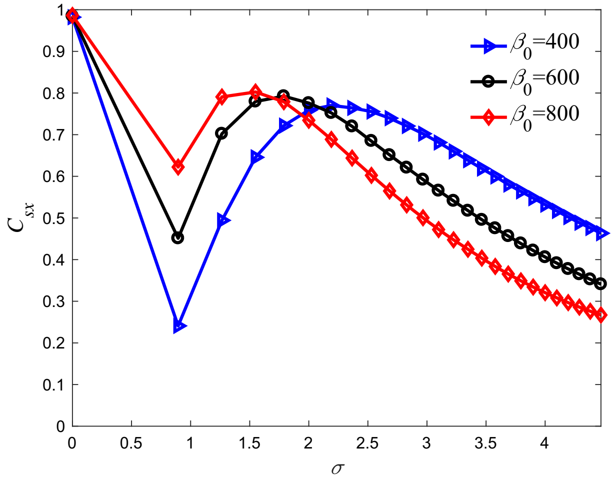

Fig. 4 The cross-correlation coefficient C_{sx} versus the noise intensity σ. Other simulation parameters are $A = 0.3$, $\gamma = 1$, $f = 1$, $\phi = 0$, $a = 1$ and $b = 1$

we only obtain an optimal value of σ in viewpoint of the whole output. In fact, we cannot assure that each segmentation achieves optimal SR. Each segmentation needs different noise intensities to achieve optimal SR.

To get optimal SR further, we adopt a more proper way to improve the cross-correlation coefficient of the whole output. Specifically, we still divide the whole signal into 5 segmentations. Then, we calculate the cross-correlation coefficient for each segmentation to find optimal noise intensity for the corresponding segmentation. Finally, we connect all optimal output segmentation in order to get the whole optimal output. Obviously, as shown in Fig. 6, we need each different noise intensity to make each segmentation to achieve its optimal SR. According to the curves in Fig. 6, Fig. 7 is depicted showing optimal SR output. Comparing

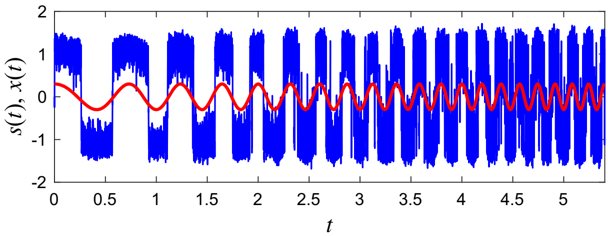

Fig. 5 Time series of the optimal SR output. The thick line in red color is the pure LFM signal. The thin line in blue color is the system output. Other simulation parameters are $A = 0.3$, $\gamma = 1$, $f = 1$, $\phi = 0$, $a = 1$, $b = 1$ and $\beta_0 = 800$

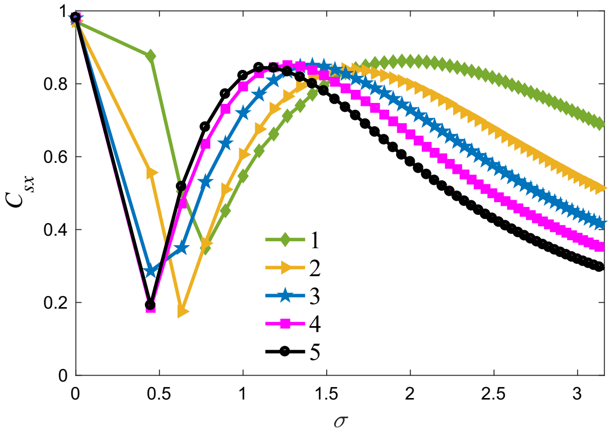

Fig. 6 The cross-correlation coefficient C_{sx} versus the noise intensity σ for different segmentations. Curves 1–5 match the segmentations 1–5, respectively. The simulation parameters are $a = 1$, $b = 1$, $\beta_0 = 800$, $A = 0.3$, $\gamma = 1$, $f = 1$ and $\phi = 0$

Fig. 7 with Fig. 5, we find that SR output in Fig. 7 does have a better performance. The C_{sx} value is 0.8343 for the time series of Fig. 5 and 0.8687 for the time series of Fig. 7, respectively. It verifies that the latter way can make SR output better. As a result, it is better to use this procedure instead of the last one.

Here are some explanations on parameter selection range and how to select. For the re-scaled coefficient, it needs to be much greater than the highest frequency of the LFM signal. The noise intensity can be optimized by the algorithm, which generally ranges from 0 to 10th the amplitude of the LFM signal. Here, system parameters are usually set to 1. The number of segmentations should depend on the length of the input, neither too much nor too less.

3 Adaptive piecewise re-scaled SR method based on APSO algorithm

In the former section, it can be found that the peaks in different segmentations are not in the same noise intensity. The selection of the noise intensity has to take a amount of time, and it may have a negative influence on the application of this method. Therefore, in order to improve the efficiency of piecewise SR method and make it quickly process the raw signal,

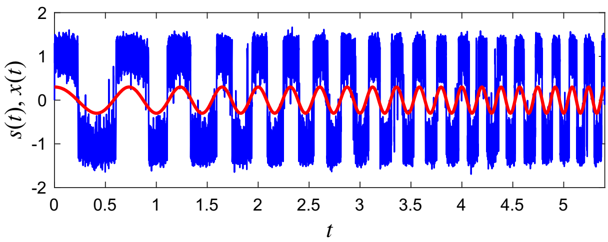

Fig. 7 Time series of the optimal SR output. The thick line in red color is the pure LFM signal. The thin line in blue color is the system output. Other simulation parameters are $A = 0.3$, $\gamma = 1$, $f = 1$, $\phi = 0$, $a = 1$, $b = 1$ and $\beta_0 = 800$

adaptive piecewise SR method is designed with the help of APSO algorithm. The specific steps are listed as follows.

Step 1 Each particle carries a possible best solution to a problem. The position of the particle represents the possible solution, while the velocity of the particle is used to update its position. The noise intensity is optimization object, and there are 40 particles in a particle swarm. After initialization, inertia weight and learning factor are set and each particle carries a random value of noise intensity. Calculate the fitness values of the particle swarm and record the current optimal position of the whole particle swarm. Before iteration, the global optimal position is equal to the current optimal position.
Step 2 Enter the iteration. In each iteration, the velocity and position of the particle are updated in adaptive way.
Step 3 After updating the velocity and position of the particle, the fitness values of the particle swarm can be calculated.
Step 4 By comparison with the previous iteration, the position corresponding to better fitness value is reserved. Sequentially, the current and global optimal position is updated.
Step 5 Repeat step 2 to step 4 until iteration time k reaches a set value $k_{\max}$.
Step 6 Repeat step 2 to step 5. The entire process of optimization is finished when the loop condition is met.
The cross-correlation coefficient is selected as the evaluation index in the course of optimization. The detailed process of piecewise SR realized by APSO algorithm is shown in Fig. 8.

Under the circumstance that the re-scaled coefficient is given, the optimal noise intensity at each segmentation is found by using APSO algorithm. Table 1 lists the results of different segmentations for different given re-scaled coefficients. To further verify the effectiveness of this method, according to these obtained results, several outputs are illustrated in Fig. 9. Obviously, the output and the pure LFM signal have a strong similarity. Compared to the pure LFM signal, the output has a relatively large amplitude, which means the LFM signal in the raw signal is enhanced. It is also worth mentioning that this method is of better convergence and higher retrieval efficiency, as shown in Fig. 10.

4 Comparison with FRFT

FRFT is one kind of common method that is often used to process the LFM signal. To further illustrate the superiority of the piecewise stochastic resonance, it will be used to compare for processing the LFM signal in the noise. The FRFT of $x(t)$ is defined as

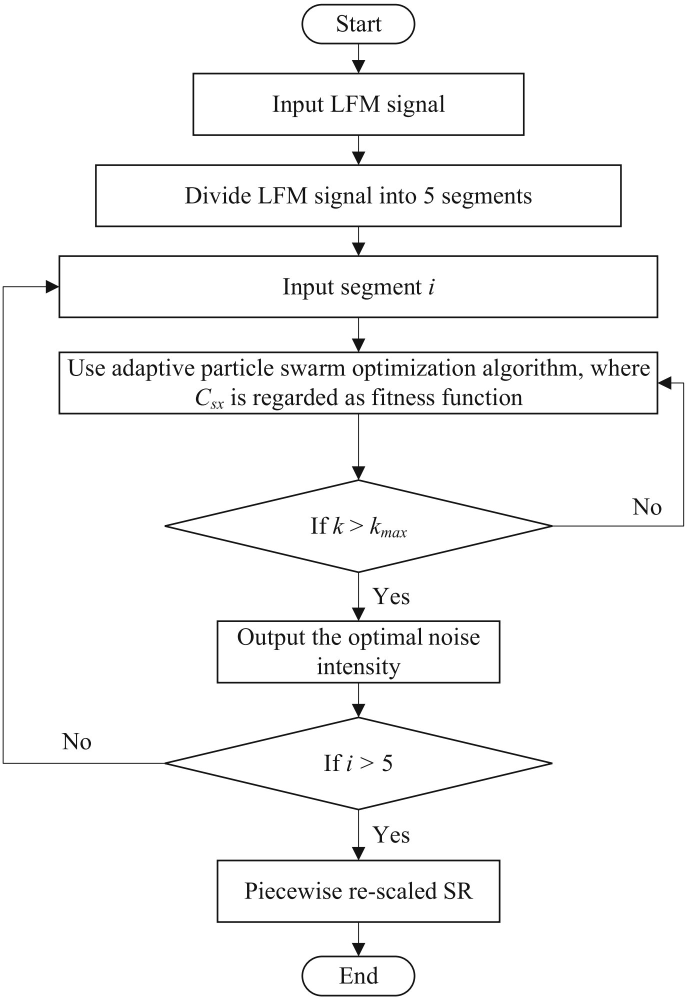

Fig. 8 The flowchart of adaptive piecewise SR method

Table 1 The optimization results for different re-scaled coefficients

	σ_1	σ_2	σ_3	σ_4	σ_5	C_{sx}
$\beta_0 = 400$	2.88374	2.69279	2.33454	1.87861	1.77162	0.8285
$\beta_0 = 600$	2.47132	1.867	1.67932	1.42767	1.50883	0.8459
$\beta_0 = 800$	1.95268	1.53957	1.35192	1.29521	6.51779	0.855

$$X_p(u) = F^p[x(t)] = \int_{-\infty}^{\infty} x(t) K_\alpha(t, u) \mathrm{d}t, \tag{8}$$

where p is the order of the FRFT and $F^p[\cdot]$ is the FRFT operator. $K_\alpha(t, u)$ is the kernel of the FRFT, represented as

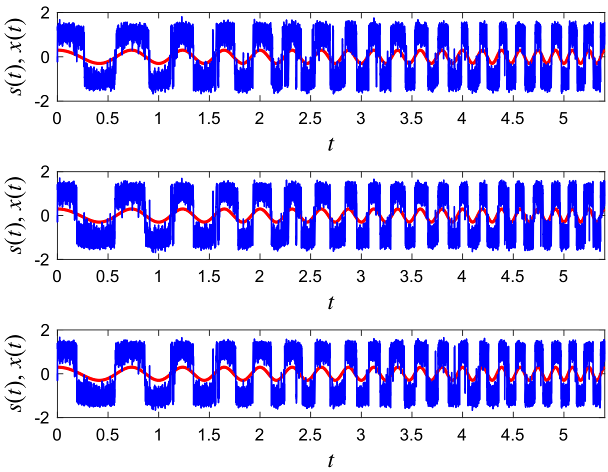

Fig. 9 The results obtained by adaptive piecewise SR method

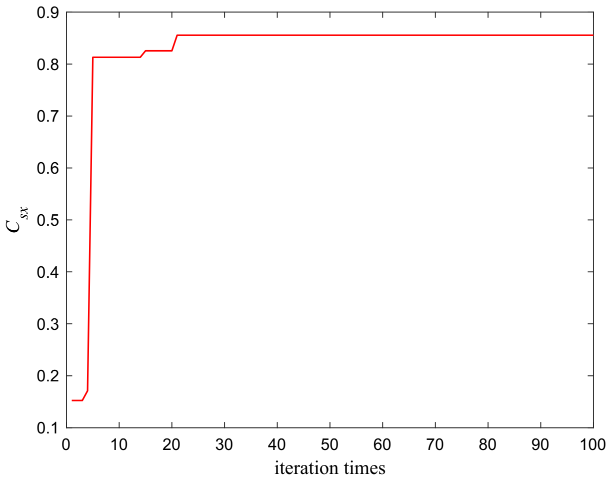

Fig. 10 The convergence curve of the adaptive piecewise SR method

$$K_\alpha(t,u)=\begin{cases}\sqrt{\frac{1-j\cot\alpha}{2\pi}}\left(\exp j\frac{t^2+u^2}{2}\cot\alpha-tu\csc\alpha\right), & \alpha\neq n\pi\\ \delta(t-u), \quad \alpha=2n\pi & \\ \delta(t+u), \quad \alpha=(2n\pm1)\pi & \end{cases} \tag{9}$$

where α is the rotation angle of axis ($\alpha=p\pi/2$). The signal $x(t)$ can be transformed into $X(u)$ in the u domain. The u domain is usually named fractional Fourier domain. For the inverse FRFT, $x(t)$ can be obtained by

$$x(t)=F^{-p}[x(t)]=\int_{-\infty}^{\infty}X(u)K_{-\alpha}(t,u)\mathrm{d}t \tag{10}$$

in which

$$K_{-\alpha}(t,u)=K_\alpha^*(t,u) \tag{11}$$

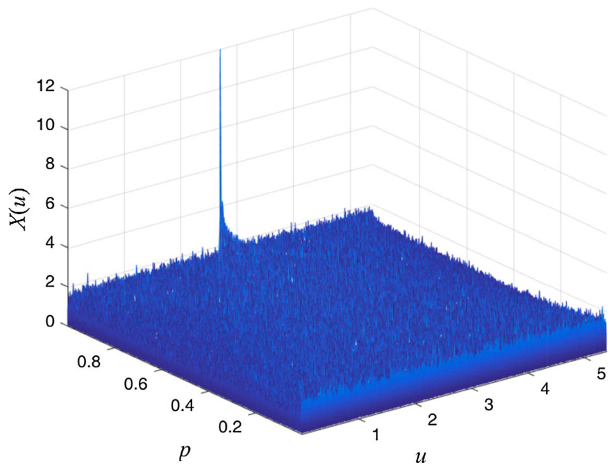

Fig. 11 The amplitude spectrum of the raw signal in the fractional Fourier domain for different fractional orders

and

$$\int_{-\infty}^{\infty} K_\alpha(t,u)K_\alpha^*(t,u')\mathrm{d}t = \delta(u-u'). \tag{12}$$

Taking the FRFT of the signal with a rotation angle α is actually a rotation of the signal. If α is selected properly, a LFM signal is able to be transformed into an impulse. This means it brings about the concentration of the energy of the LFM signal. However, if noise is processed directly by the FRFT, the noise energy is still distributed symmetrically and will not be focused in the fractional Fourier domain, whichever the value of α is chosen. According to this property, therefore, a method has been proposed for separating an LFM signal from the raw signal. The procedure of such method is as follows.

Step 1 Plot the FRFT outputs with different fractional orders and obtain optimal order p_0.

Step 2 Take the FRFT of the raw signal and then the signal is rotated with an optimal angle α_0 ($\alpha_0 = p_0\pi/2$). The rotated signal is represented as

$$X_{p_0}(u) = S_{p_0}(u) + N_{p_0}(u) \tag{13}$$

in which $S_{p_0}(u)$ and $N_{p_0}(u)$ are the FRFTs of the LFM signal and noise. The output therefore can be obtained in the u domain, and obvious peak will emerge.

Step 3 According to this peak, a mask operation is adopted and then get

$$X_{p_0}{}'(u) = X_{p_0}(u)M(u) = S_{p_0}(u)M(u) + N_{p_0}(u)M(u) \tag{14}$$

where $M(u)$ is a narrow filter with central frequency u_0. If the bandwidth is properly selected, most energy of the noise will be removed. On the contrary, the energy of the LFM signal will be reserved.

Step 4 Take the inverse FRFT and then the signal after filtering will be rotated back to the time domain.

According to this method, Fig. 11 is firstly plotted. A distinct peak appears in the (p, u) plane, and thus, we get an optimal order p_0. After the FRFT with its optimal order p_0, raw signal $r(t)$ is transformed into $X(u)$, as shown in Fig. 12. Obviously, the LFM signal contained in the raw signal is turned into an impulse in u domain, but the noise is not. Then select the narrow filter with a proper bandwidth and conduct the mask operation. Finally,

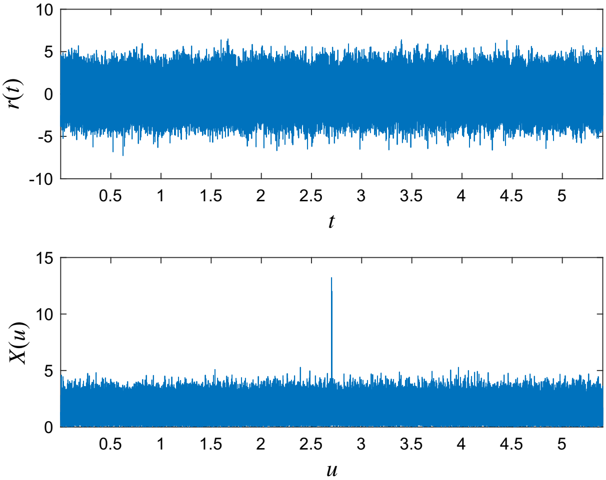

Fig. 12 The time series of the raw signal and its amplitude spectrum in the fractional Fourier domain for optimal order

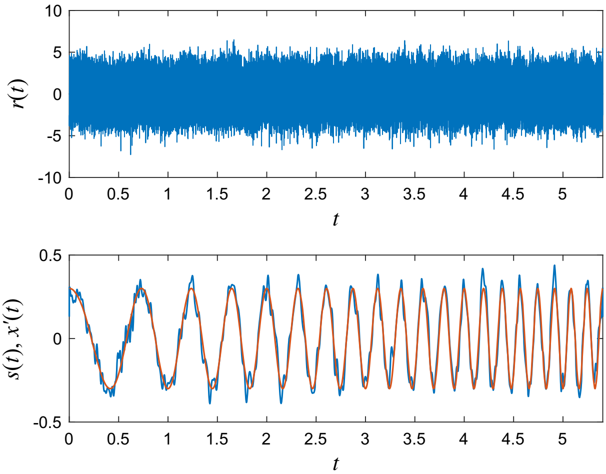

Fig. 13 The time series of the raw signal, the output processed by FRFT and the result after fitting it

take the inverse FRFT and get the signal $x'(t)$ shown in Fig. 13. The signal after a series of processes $x'(t)$ is quite similar to the pure LFM signal $s(t)$. Its cross-correlation coefficient reaches 0.97. In other words, the LFM signal has been separated from the noise.

To make comparison between this method and piecewise SR in signal waveform, we use nlinfit function to best fit the data of Fig. 7 by MATLAB. We thus get Fig. 14, and obviously the waveform $y(t)$ after fitting is also fairly similar to the pure LFM signal. Its cross-correlation coefficient reaches 0.989. As a result, by comparing between these two methods in waveform, we find that piecewise SR has the same separation performance as the FRFT. However, by comparing their amplitudes, we find that the LFM signal is obviously enhanced by using piecewise SR. It means the energy of the noise is transferred to the LFM signal. To further illustrate this point, we take the FRFT of the signal of Fig. 7 and then get

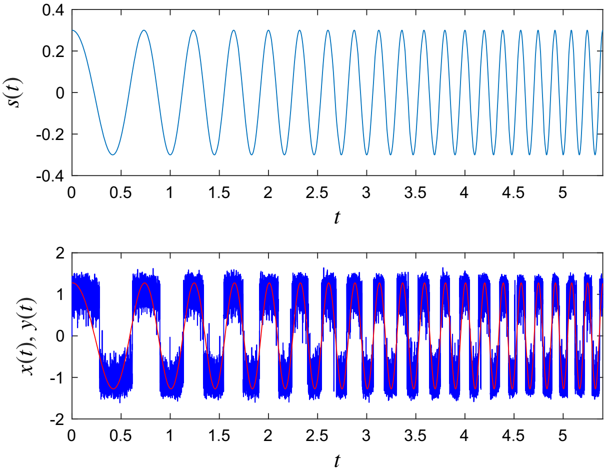

Fig. 14 The pure LFM signal, the optimal SR output and the result after fitting it

Fig. 15 The amplitude spectrum of the optimal output in the fractional Fourier domain for different fractional orders

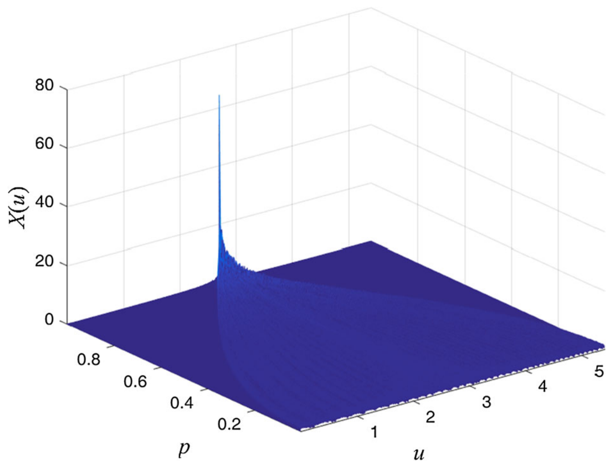

Fig. 15. Next according to the optimal order, we obtain Fig. 16. Comparing Figs. 16 and 12, we can observe that the energy of noise is almost used to enhance the LFM signal. In short, these two methods both have a favorable separation performance, but the proposed method is able to greatly enhance the LFM signal rather than weaken it.

5 Conclusion

In the present work, we study SR in the LFM signal excited system. Since the frequency is time modulated, it is difficult to realize SR by the traditional SR theory. To solve this technical problem, piecewise SR method is proposed.

To make the signal satisfy the precondition for traditional SR theory, we introduce the piecewise idea to process the whole signal. The method is piecewise re-scaled SR method. By a general transformation for each signal segmentation, we make SR to occur for each

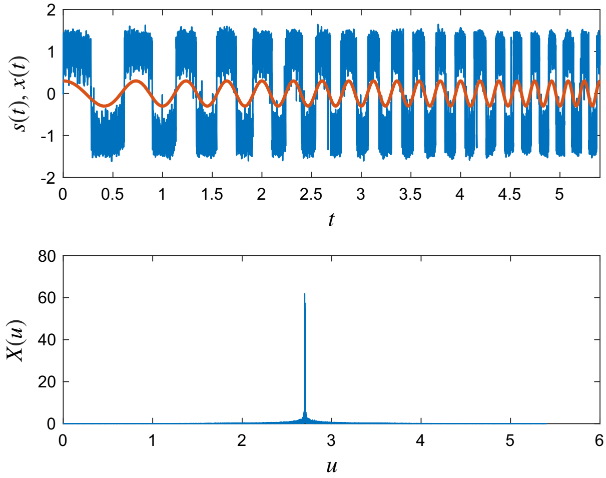

Fig. 16 The time series of the optimal SR output and its amplitude spectrum in the fractional Fourier domain for optimal order

output segmentation. Then, SR is realized for the whole output. Both the cross-correlation coefficient index and the input/output synchronization are used to show SR performances. In addition, adaptive piecewise re-scaled SR is proposed and it can quickly achieve optimal SR. By some numerical examples and comparison with FRFT, the effectiveness of the two methods is verified. However, in signal enhancement, the proposed method has a better behavior.

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