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An Essay on the Origins and Development of Nonlinear Dynamics, Chaos and Complex Systems

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ABSTRACT

We can find dynamics in every field of science, including economics, chemical reactions, physiology or neurodynamics, showing its intrinsic interdisciplinarity. The interactions between the parts of a system and their feedback mechanisms constitute a source of nonlinearity and complexity, which added to the sensitive dependence on initial conditions, hallmark of chaotic behavior, lead to a crucial change of perspective with serious consequences in the understanding of science. Relevant problems such as the three-body problem in celestial mechanics, turbulence in fluid dynamics, irreversibility in statistical physics; or the logistic equation in population dynamics, have been at the origins of this fascinating field of nonlinear dynamics, chaos, and complex systems. A knowledge of its origins, as well as the many schools of mathematics and physics that have contributed to its development, allow us to better understand the discipline and the breadth of its many applications to science.

1. Introduction

Nonlinear dynamics is the discipline that aims to study nonlinear dynamical systems, which are those systems defined by one or more variables evolving with time where the response is not proportional to the stimulus. Chaos is one of the three kinds of motion, in addition to the periodic and quasi-periodic motions. Naturally, there are as many dynamical systems as there are variables that have a temporal evolution, which gives us an idea of the interdisciplinary nature and scope of nonlinear dynamics [1-4].

Many of the ideas and concepts of complexity, such as a science of complex systems, pose a real challenge for the integration of various disciplines, among which we must point out nonlinear dynamics and chaos theory, statistical physics, stochastic processes, information theory, network theory, engineering science, life sciences, and computer sciences. This listing is naturally not complete, but it gives an idea of the challenge behind the idea of complexity. This intended goal means more than the idea of crossing disciplinary boundaries, but rather to integrate disciplines in a common background.

Much has been discussed in recent years about a fruitful dialogue between different scientific disciplines, not only to solve old problems, but also as a source of inspiration for new problems. For the study of complexity this is one of the fundamental elements, since its object of study covers problems related to both the so-called hard and soft sciences. Complex systems exist in biology, chemistry, physics, sociology, economics, etc. In any case, the true dialogue between disciplines so necessary for the advancement of knowledge of complex systems in particular, and of science in general, is still lacking.

Different paths that have led to the understanding of chaos as we understand it today. Among them, I like to point out: (1) The logistic map and population dynamics (2) Nonlinear oscillators (3) The three-body problem in celestial mechanics (4) Turbulence in fluid dynamics and (5) Irreversibility in statistical mechanics. All of them will be discussed throughout this article.

2. Nonlinear Dynamics and Deterministic Chaos

As previously discussed, dynamics is the science that studies the variation in time of different variables, that is, its motion. Basically, there are three types of motion: stationary and equilibrium; periodic and quasi-periodic; and finally chaotic motion. Considering the notion of motion in a broad sense, it is easy to understand that we can find dynamical systems in any scientific

discipline. That is why it is customary to say that one of the characteristics of nonlinear dynamics is its interdisciplinarity, since with its methods we can approach the study of many different phenomena that evolve over time.
We use the term "nonlinear" to logically contrast it with the term "linear", since the linear approach is the one traditionally used in science due to its mathematical simplicity. The linear approach implies the assumption of properties such as: (1) Proportionality: small causes cause small effects (2) Additivity: the whole is equal to the sum of its parts (3) Replication: the same action under the same conditions produces the same result and (4) clear relationships between cause and effect: it is enough to know a little about the behavior of a system to fully know it.

However, when nature's relationships are not linear, it leads us to very different situations. A proportional relationship between two variables x and y, where $y = kx$, indicates a linear relationship. Therefore, any relationship between two variables that does not respond to a proportional relationship like the previous one will be nonlinear. It is easy to figure out that most dynamical systems are nonlinear.

When there are relationships of nonlinearity, there can be chaotic behavior that has the following properties: (1) There is no proportionality: small causes can cause large effects (2) Emergence: additivity does not exist, so the whole is greater than the sum of its parts (3) Sensitive dependence on initial conditions: which can make that the same experiment can never be reproduced exactly; and finally (4) Nonlinearity that can generate instabilities, discontinuities and unpredictability, which requires flexibility, adaptability, dynamic change, innovation and reaction capacity.

Possibly one of the deepest ideas about the nature of what is known as chaotic behavior is the idea of sensitive dependence on the initial conditions. That is, trajectories of a chaotic system move away from each other as time progresses when they start from very close initial points. This fact has very drastic consequences on the predictability of a system.

From this viewpoint, it is somehow surprising to read the following sentence from the Chapter XIV of *The Origin of Species* (1859) [5] by Charles Darwin:

"More individuals are born than can possibly survive. A grain in the balance will determine which individuals shall live and which will shall die, which variety or species shall increase in number, and which shall decrease, or finally become extinct",

that in a certain sense shows already the true notion of sensitive dependence on initial conditions.

In this regard, it is also interesting to bring up a famous rhyme traditionally associated with Benjamin Franklin, although antecedents of the same idea date back to the 15th century, and which is known as *"For Want of a Nail "* as shown in Fig. 1.

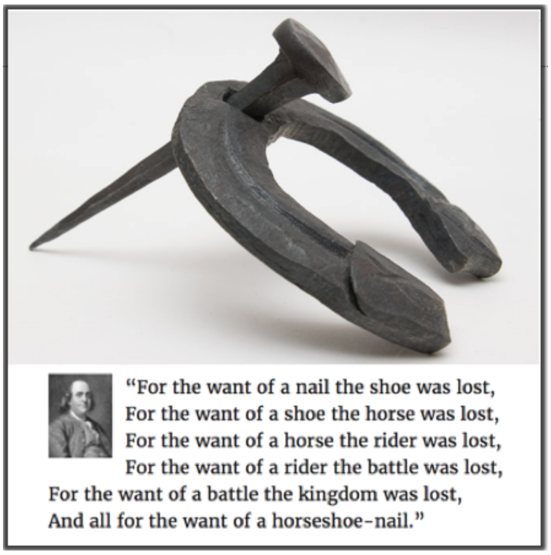

Fig. 1: The rhyme *"For Want of a Nail..."* offers an intuitive and poetic image of the idea of sensitive dependence on initial conditions, which is the hallmark of chaos.

We can define chaotic behavior or chaos as a type of motion that is derived from deterministic temporal dynamics of simple systems that can in fact be described in terms of few variables and whose fundamental characteristics are: (1) Being irregular in time, and given its nonlinear character, of course, cannot be the superposition of periodic motions, being in fact of aperiodic nature and bounded (2) Be unpredictable in the long term and very sensitive on the initial conditions and (3) Be complex, but ordered in the phase space, presenting a geometry of a fractal nature. If we compare the chaotic motion with the regular motion, we can say that the latter is repetitive, periodic, predictable and with a simple geometry, while the former is irregular, unpredictable and with a complicated geometry.

There are different types of chaotic motions. It is fundamentally called permanent chaos when once a dynamical system finds this state it remains in it forever. On the other hand, it is called transient chaos when this chaotic behavior occurs only in a certain period of time and the system subsequently behaves differently. Furthermore, dynamical systems generally distinguish between dissipative and conservative based on whether or not they conserve energy. Well, for dissipative systems permanent chaos occurs in what is called a chaotic attractor in the phase space. However, in the case of transient chaos, chaotic transients occur in a fractal set. In the conservative case, on the one hand, permanent chaos occurs in bounded regions of the phase space and transient chaos is associated, for example, with the phenomenon of chaotic scattering that occurs in numerous physical phenomena, giving rise to very complex fractal structures. These concepts will be explained in more detail throughout the article.

Dynamical systems are usually classified as discrete and continuous depending on whether time is measured discretely or continuously. A paradigm for discrete dynamical systems is the logistic map, defined as

$$x_{n+1} = rx_n(1 - x_n),$$

which is an iterative equation where the index n indicates an iteration that is linked to the discrete way of measuring time. Figure 2 shows a Feigenbaum bifurcation diagram corresponding to the logistic map, where the final state of the system is displayed as a function of the variation of parameter r.

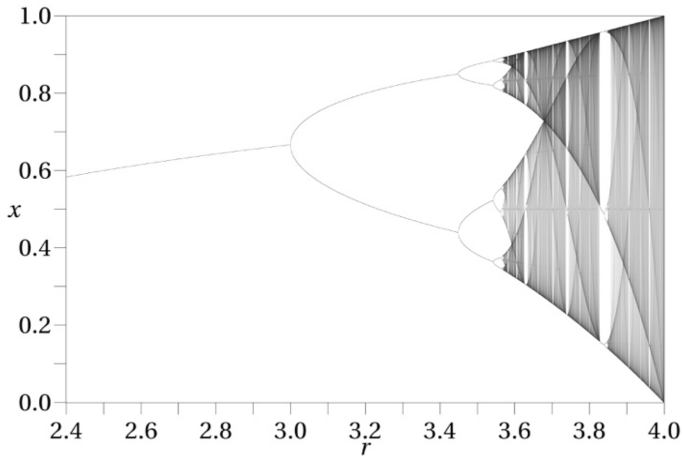

Fig. 2: Feigenbaum bifurcation diagram corresponding to the logistic map. It indicates how the final state of the system varies depending on the value of parameter r.

A paradigm for continuous systems is the simple pendulum (Fig. 3). It consists of a body of mass m that hangs on a cord that is in principle inextensible and of negligible mass, and whose suspension point moves periodically.

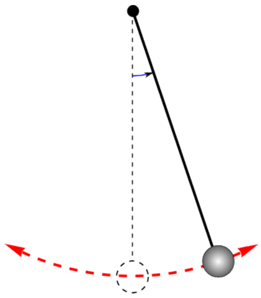

Fig. 3. Oscillatory motion of a pendulum

In this system, time is measured continuously, and therefore it can be modeled, once normalized, using a differential equation such as

$$\ddot{x} + \mu\dot{x} + \sin x = F\cos\omega t\ .$$

This equation contains in addition to the inertia term (the second derivative of the position), the friction of intensity μ which is proportional to the velocity, the nonlinear sinusoidal term and an external periodic forcing of amplitude *F* and frequency ω. If we consider the friction with the air and assuming that the suspension point remains fixed, then the motion will gradually dampen until it stops in its stable equilibrium position. When the suspension point moves periodically, it has the effect of introducing energy into the system, causing oscillations to be maintained. However, it is also possible to give rise to another type of motion of an irregular and unrepeatable nature on a periodic basis, which is chaotic motion.

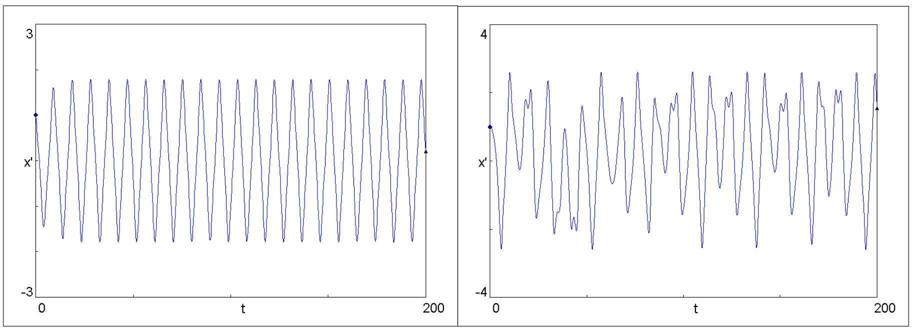

Fig. 4. Evolution of velocity over time for periodic and chaotic motions.

Figure 4. indicates the time evolution of the velocity of a pendulum. In one of them clearly the periodic nature of the oscillations can be observed, that is, after a certain period of time the same motion is repeated. In the other figure, however, an irregular behavior is shown, which turns out to be chaotic, where it can be observed that the same type of motion is not

reproduced after any period of time. This is precisely one of the characteristics of chaotic motion, its lack of periodicity.

A simple example of periodic system is the mass-spring system formed by a body that is attached by a spring to a wall (Fig. 5). If the displacement with respect to the equilibrium position is very small, then the spring recovery force is proportional (linear) to the displacement, so that the result of motion is regular, oscillatory, and periodic.

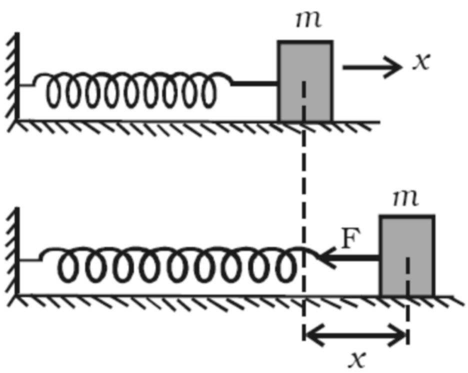

Fig. 5. Oscillatory motion of a system formed by a body of mass *m* attached to a spring.

When the deformation is greater, the spring recovery force is not linear, leading to irregular spring responses. In this situation, the resulting motions can be very irregular, and may be chaotic in nature where there are no regularities or periodicities and where the long-term predictability is lost.

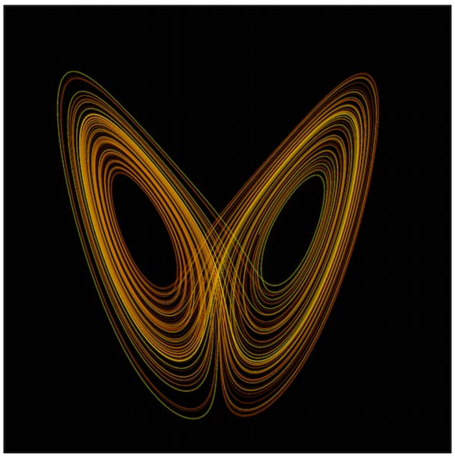

Fig. 6: The figure represents the chaotic attractor of the Lorenz system.

Perhaps one of the more well-known chaotic systems is the Lorenz system, which at the same time is one of the most studied chaotic systems. Figure 6 shows a chaotic attractor of the Lorenz system in phase space. It was introduced by the meteorologist Edward Lorenz to study the thermal convection in a fluid and by means of computer numerical simulations he was able to observe the property of sensitive dependence on initial conditions, the hallmark of chaotic behavior.

Figure 7 shows the idea of sensitive dependence on initial conditions in the Lorenz chaotic system. The figure shows the temporal evolution in phase space of two orbits (one red and one blue) whose initial conditions are very close. After a certain time, approximately 24 time-units, the corresponding orbits start to drift apart, turning out to be very different at long times.

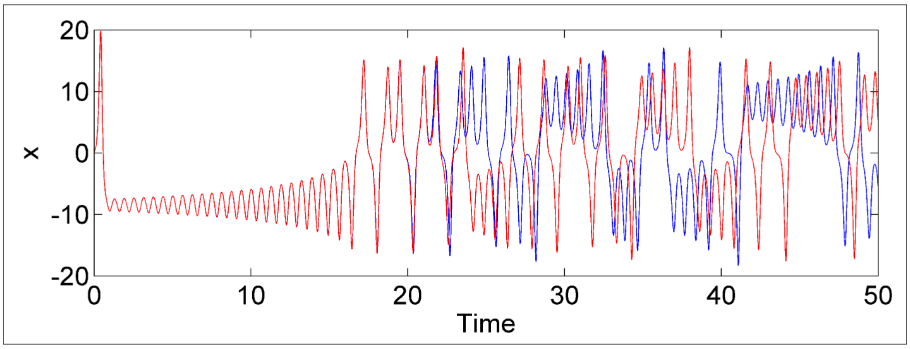

Fig. 7. Time evolution in phase space of two orbits of the Lorenz system initially very close, where the property of sensitive dependence on initial conditions is shown.

A very important tool in nonlinear dynamics is the geometric notion of phase space. The notion of phase space [6] is attributed to the American physicist Josiah Willard Gibbs (1839-1903), who was one of the pioneers of kinetic theory and is also considered one of the founding fathers of statistical mechanics, a term that he also coined. The concept of phase space plays a crucial role in nonlinear dynamics, from whose analysis we can obtain much information about a given dynamical system.

Studying the phase space of a given dynamical system allows complex fractal structures to be obtained whose physical consequences are reflected in uncertainty when determining the subsequent state of the system (Fig. 8).

3. A Historical Overview of Nonlinear Dynamics

Throughout the 19th century, certain limitations appeared around the myth of determinism. On the one hand, it is essential to have a complete knowledge of the initial conditions of the problem. On the other hand, notable difficulties arose in solving the dynamics of a physical system made up of a large number of particles. The latter led to the introduction of concepts related to probability theory in the study of the physical laws of systems made up of many particles, such as gases, liquids and solids, giving rise to the birth of statistical mechanics. The founding fathers of the discipline include Ludwig Boltzmann (1844-1906), Josiah Willard Gibbs (1839-1903), and James Clerk Maxwell (1831-1879).

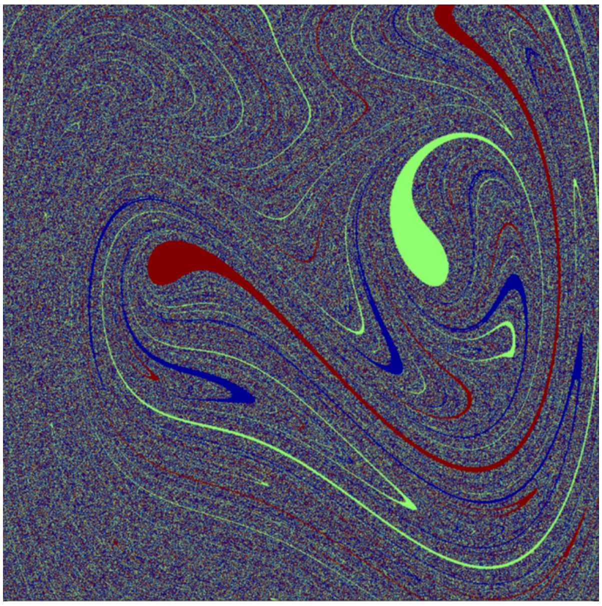

Fig. 8. Fractal structures in the phase space of a chaotic nonlinear oscillator. The variables of the phase space are the position on the x axis and the velocity on the y axis.

Scottish physicist James Clerk Maxwell (1831-1879), is fundamentally known for having unified the laws of electricity and magnetism. However, his contributions to physics have been among the most prolific in history. Among his great scientific work, it is important to mention that he is considered the father of automatics and statistical mechanics. However, the role he played in the development of modern chaos theory is largely unknown.

Precisely in one of his writings: *Does the progress of physical science tend to give any advantage to the opinion of necessity (or determinism) over that of the contingency of events and the freedom of the will?* from a lecture he gave at Cambridge on February 11, 1873 are the following excerpts showing to what extent Maxwell was familiar with the idea of sensitive dependence on initial conditions, of which we have spoken earlier.

"Much light may be thrown on some of these questions by the consideration of stability and instability. When the state of things is such that an infinitely

small variation of the present state will alter only by an infinitely small quantity the state at some future time, the condition of the system, whether at rest or in motion, is said to be stable; but when an infinitely small variation in the present state may bring about a finite difference in the state of the system in a finite time, the condition of the system is said to be unstable. It is manifest that the existence of unstable conditions renders impossible the prediction of future events, if our knowledge of the present state is only approximate, and not accurate."

Due to the enormous consequences on determinism in physics that quantum mechanics has brought about through Heisenberg uncertainty principle, the idea of indeterminism has been directly related to quantum mechanics. This has led somehow to consider classical mechanics as completely deterministic and predictable, which is not entirely true [7].

It is fascinating to corroborate that the idea of sensitive dependence on initial conditions was considered in detail by the German physicist Max Born (1882-1970), Nobel Prize in Physics in 1954, in an article entitled *Is Classical Mechanics in fact deterministic?* [8]. In it he presented a study of a two-dimensional Lorentz gas initially proposed by the Dutch physicist Hendrik A. Lorentz (1853-1928) in 1905 as a model for the study of electrical conductivity in metals. In this model, a particle moves in a plane that is full of hard spheres and collides with them so that a small change in the initial conditions will significantly alter the trajectory of the particle. This fact led Born to conclude that determinism traditionally related to classical mechanics is not real, since it is not possible to know with infinite precision the initial conditions of a physical experiment.

Furthermore, in the lecture [9] he gave on the occasion of the awarding of the Nobel Prize in 1954 the following words appear:

"Newtonian mechanics is deterministic in the following sense: If the initial state (positions and velocities of all particles) of a system is accurately given, then the state at any other time (earlier or later) can be calculated from the laws of mechanics. All the other branches of classical physics have been built up according to this model. Mechanical determinism gradually became a kind of article of faith: the world as a machine, an automaton. As far as I can see, this idea has no forerunners in ancient and medieval philosophy. The idea is a product of the immense success of Newtonian mechanics, particularly in astronomy. In the 19th century it became a basic philosophical principle for the whole of exact science. I asked myself whether this was really justified. Can absolute predictions really be made for all time on the basis of the classical equations of motion? It can easily be seen, by simple examples, that this is only the case when the possibility of

absolutely exact measurement (of position, velocity, or other quantities) is assumed. Let us think of a particle moving without friction on a straight line between two end-points (walls), at which it experiences completely elastic recoil. It moves with constant speed equal to its initial speed v_0 *backwards and forwards, and it can be stated exactly where it will be at a given time provided that* v_0 *is accurately known. But if a small inaccuracy* Δv_0 *is allowed, then the inaccuracy of prediction of the position at time t is* $t\Delta v_0$ *which increases with t. If one waits long enough until time* $t_c = l/\Delta v_0$ *where l is the distance between the elastic walls, the inaccuracy* Δx *will have become equal to the whole space l. Thus it is impossible to forecast anything about the position at a time which is later than* t_c*. Thus determinism lapses completely into indeterminism as soon as the slightest inaccuracy in the data on velocity is permitted."*

The American physicist Richard Feynman (1918-1988), who won the Nobel Prize for Physics in 1965 (Fig. 9), makes similar reflections in his well-known book *Lectures in Physics* [10], where he explains that indeterminism does not belong exclusively to quantum mechanics, it is a basic property of many physical systems.

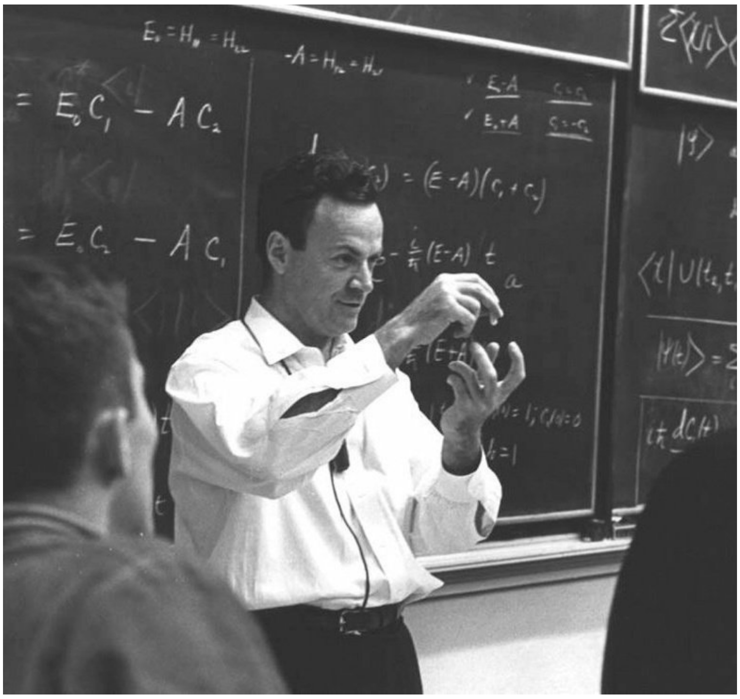

Fig. 9. Richard Feynman (1918-1988), Nobel Prize in Physics in 1965.

In section 38-6, entitled "Philosophical Implications", in the first volume of his *Lectures in Physics*, a masterful description of indeterminism in classical mechanics is made. The fundamental idea is the uncertainty in accurately setting initial conditions to predict the final state of a physical system. Finally affirming: "Because in classical mechanics there was already indeterminism from a practical point of view".

3.1 Poincaré, the Three-Body Problem and the Birth of Chaos

To understand the three-body problem, we go back to the beginnings of modern science with Isaac Newton's works on the gravitational field and the universal law of gravitation. The so-called two-body problem basically consists of analyzing the motion of a system formed by two bodies that attract each other under the action of gravitational forces. Newton solves the problem by reducing the motion of the two bodies to the motion of each of them around the so-called center of mass, which is a point whose mass is the total mass of the system.

Later, an attempt was made to solve the three-body problem, which can be formulated in a simple way: Let 3 bodies of arbitrary masses m_1, m_2 and m_3 be mutually attracted by Newton's law of gravitation. Assuming that they can move freely in a three-dimensional space and with arbitrary initial conditions, determine the evolution of the motion.

Despite the simplicity of its formulation, its resolution has caused real headaches for many scientists. Among them we may highlight Isaac Newton (1642-1727), Alexis Clairaut (1713-1765), Leonhard Euler (1707-1783), Pierre-Simon Laplace (1749-1827), Joseph-Louis Lagrange (1736-1813), Carl Jacobi (1804-1851), George Hill (1838-1914) and Henri Poincaré (1854-1912).

It is precisely the latter who wrote a famous memoir in 1889 on *the three-body problem and the equations of dynamics*, after winning the prize of the contest on the stability of the Solar System that had been summoned by King Oscar II of Sweden and Norway on the occasion of his 60th anniversary. This competition [11] had been proposed by the Swedish mathematician Gösta Mittag-Leffler, who had received it from the German mathematician Karl Weiertrass, who had been his teacher, the idea that the contestants write an original work facing one of four questions. One of Weiertrass's four questions had to do with Celestial Mechanics. The question was born out of a suggestion formulated by the mathematician Peter Gustav Lejeune Dirichlet at the University of Göttingen, who in 1858 had told his student Leopold Kronecker that he had discovered a new method of solving certain differential equations and pointed out that by applying them to the equations of celestial mechanics he could prove with all rigor that the solar system was stable. The committee that evaluated this competition was made up of mathematicians Karl Weiertrass, Frenchman Charles Hermite and Swedish Gösta Mittag-Leffler.

Subsequently, in 1892, Poincaré published his great work *Les Méthodes nouvelles de la Mécanique Céleste* (Fig. 10) in three volumes where numerous new concepts appear that have given rise to the development of the theory of dynamical systems, as mathematicians usually call it or nonlinear dynamics, a term more used by physicists, as well as other mathematical disciplines such as topology. Poincaré is considered to be one of the fathers of chaos theory, as many fundamental ideas of the theory are contained in this book.

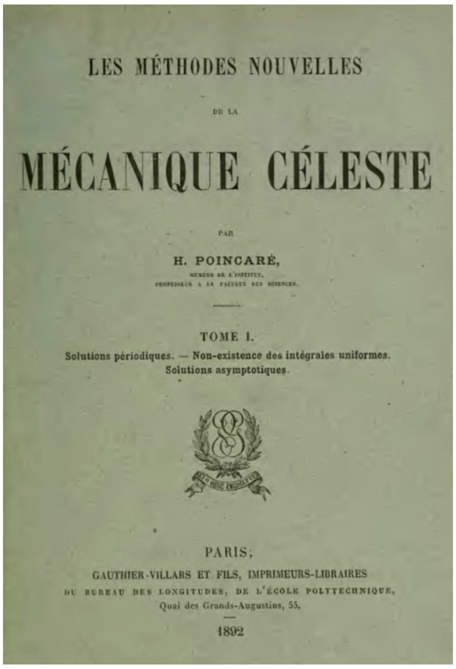

Fig. 10. *Les Méthodes Nouvelles de la Mécanique Céleste* was published by Henri Poincaré in 1892.

The general three-body problem is of enormous difficulty and only in recent years notable advances have been made, without being definitive. However, there is a case that is called restricted, circular and plane, which is the one that has been studied by many of the scientists to whom I have previously referred. Fundamentally, it is considered that the system is not made up of any three masses, but one of them is considered much larger than the others and the third of them is of negligible mass compared to the rest. The analogy certainly comes from considering systems like the Sun, Earth, and

Moon, or Earth, Moon, and a satellite, where the approximation of moving in a plane is also correct. In these circumstances and with an appropriate reference system, the equations of motion can be found without difficulty, from which a potential is derived that gives us an idea of the equilibrium positions in which a third body can be found. These are five equilibrium positions that Lagrange found, which is why they are currently known as the Lagrange points (Fig. 11).

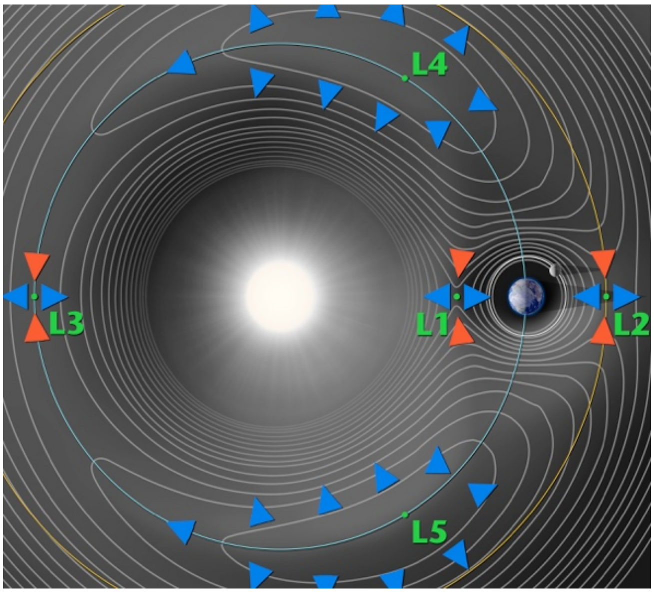

Fig. 11. The figure shows the equipotential curves of the restricted three-body problem, in this case, Sun-Earth-Moon where the five Lagrange points are illustrated.

Knowledge of the Lagrange points is very useful. In fact, at point L1 is the Solar and Heliospheric Observatory (SOHO), which is a space probe to study the Sun. At Lagrange point L2, the Wilkinson Microwave Anisotropy Probe (WMAP) was positioned to study radiation from microwave background of the universe, getting it to stay in place with minimal fuel consumption, always keeping its sensors pointed away from the Earth and the Sun. The James Webb Space Telescope (JWST) is planned to be launched in 2021, which is a developing space observatory that will study the sky in infrared frequency, and that will orbit around the L2 Lagrange point.

As pointed out above, Poincaré did not approach the three-body problem in a general way, but focused on studying what is known as the "restricted three-body problem", which is a particular case in which it is considered that one of the masses is very small compared to the others. In this study he found what he called doubly asymptotic or homoclinic orbits, which are characterized by having a homoclinic point in the phase space. The presence of one of these points have very serious implications on the dynamic complexity of the system. After studying the problem, Poincaré wrote:

"One will be struck by the complexity of this picture that I do not even dare to sketch. Nothing is more appropriate to give us an idea of the intricateness of the three-body problem and in general all problems of dynamics…"

And it is that when trying to solve this problem he created a method or a geometric approximation by means of which he glimpsed that this problem had a very complex dynamics that is basically what we now call deterministic chaos.

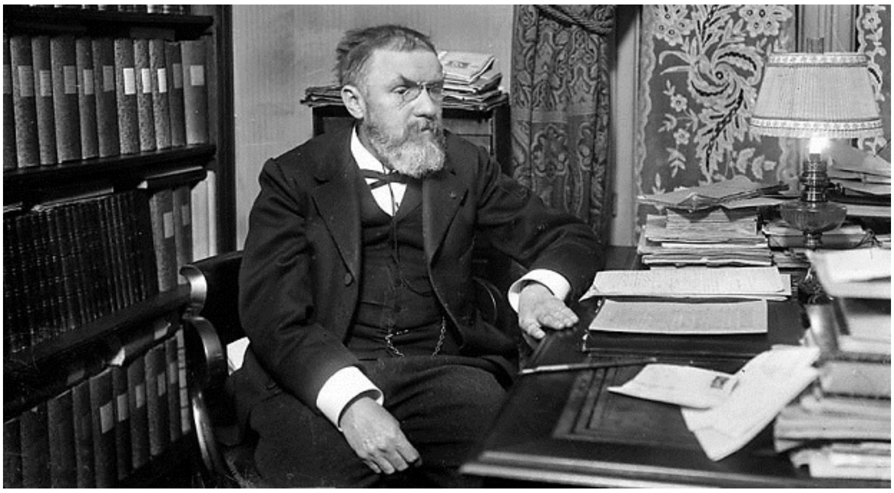

Fig. 12. French mathematician and physicist Henri Poincaré (1854-1912)

The influence of Poincaré (Fig. 12) on the development of Hamiltonian systems is enormous and in this sense it is interesting to mention that his witness was taken by the American mathematician George David Birkhoff (1884-1944), who coined the term dynamical systems, since in turn it had an enormous influence on Edward Lorenz who would rediscover the sensitive dependence on initial conditions in the middle of the 20th century.

Within this stream of thought, and in the American context, it is necessary to mention the mathematician Steven Smale (Fig. 13), deserving of the Fields medal in 1966 for his great contributions to the theory of dynamical systems. It is precisely to him that the concept of Smale horseshoe is due, which was an important step in understanding the relationship between the existence of a homoclinic point and the notion of deterministic chaos, through the simple idea of symbolic dynamics using the so-called Bernoulli shift map.

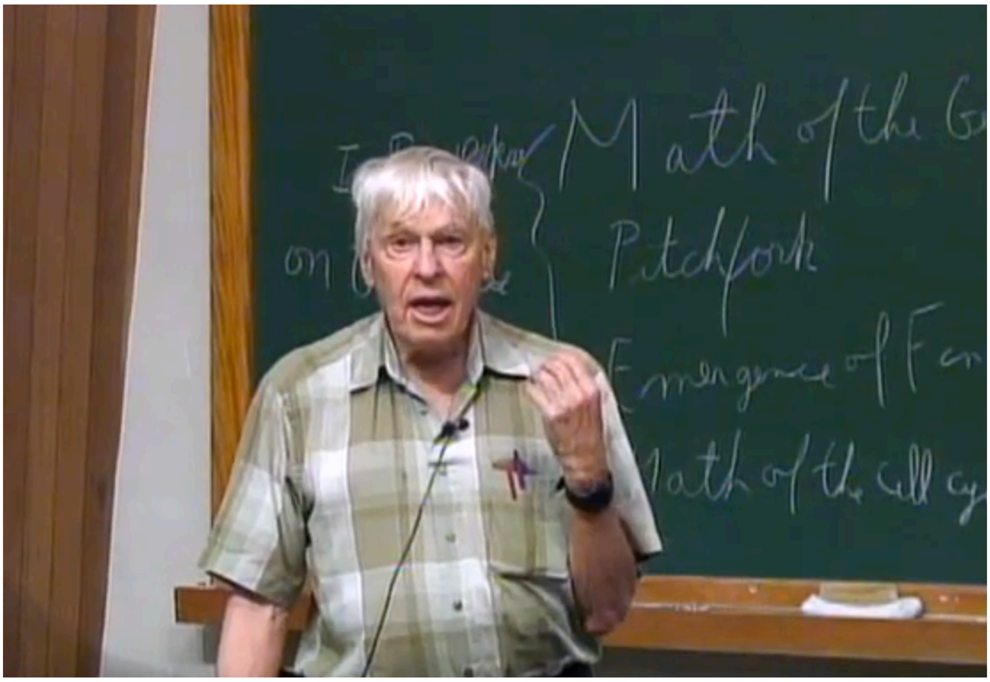

Fig. 13. Steve Smale, Fields Medal 1966.

As for the Russian tradition, we must go back to Alexander M. Lyapunov (1857-1918), who had been a doctoral student of the famous mathematician Pafnuti L. Chebychev (1821-1894), and whose thesis on the stability of motion has exerted an enormous influence on Physics. From Lyapunov we have inherited concepts such as the stability of dynamical systems and also Lyapunov's useful exponents, which help us to characterize when a given dynamical system is chaotic or not.

One of the main schools within the Russian tradition is that of Leonid I. Mandelstam (1879-1944), continued by his disciples Alexander A. Andronov (1901-1952) and Lev S. Pontryagin (1908-1988). Another key school within this same tradition is that of Andrei N. Kolmogorov (1903-1987). All of them developed new methods and made notable contributions to the construction of nonlinear dynamics as we know it today.

In the year 1954, at the International Congress of Mathematics that took place in Amsterdam, Kolmogorov enunciated a theorem for Hamiltonian systems that was subsequently proved by his student Vladimir I. Arnold and by the German Jürgen Moser (1928-1999), who has turned out to be of considerable importance. This theorem is currently known as the KAM theorem (Kolmogorov-Arnold-Moser) [12] and it has to do with the problem of the stability of invariant tori in the integrable systems of Hamiltonian mechanics under the action of small perturbations.

This work, in fact, naturally links with Poincaré's pioneering works on celestial mechanics, since he had brought out the idea of the complexity of orbits in the three-body problem, and the KAM theorem can be considered as a culmination of these ideas. As we have already seen, the stability of the solar system is a problem of special importance in celestial mechanics and the KAM theorem shows that under certain conditions these orbits remain confined in certain regions.

3.2 Complexity in Fluid Motion

The phenomenon of turbulence in fluid motion is one of the most spectacular cases of chaotic behavior. Although the fundamental equations of fluid motion, the Navier-Stokes equations, have been known since the end of the 19th century, it should be remembered that the form of their solutions in turbulent regime is not yet known.

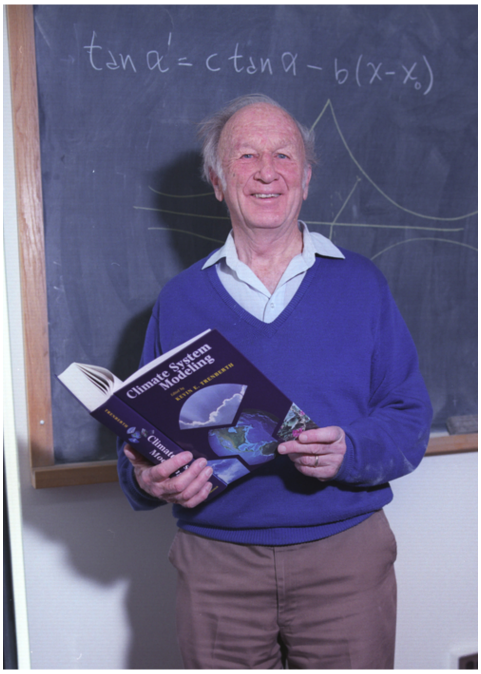

Fig. 14. Edward N. Lorenz (1917–2008)

In 1963 the meteorologist at the Massachusetts Institute of Technology (MIT) Edward N. Lorenz (Fig. 14) developed a model of three ordinary differential equations to describe the motion of a fluid under the action of a thermal gradient. When it came to finding numerical solutions with the help of a computer, he again encountered the phenomenon of sensitive dependence on initial conditions. That is, the system was inherently unpredictable, such that small variations in determining the initial conditions led to drastically different solutions.

At the time, very few gave importance to this fact, perhaps because the results of Lorenz's work were published with a somewhat cryptic title, *"Deterministic Nonperiodic Flow",* [13] in a meteorology journal and went unnoticed by many scientists.

The theory of the Russian physicist Lev D. Landau, and the German Eberhard Hopf that proposed the existence of an infinite set of incommensurable frequencies to explain the turbulence, was surpassed in the 1970s by the theoretical contributions of David Ruelle and Floris Takens, who introduced in 1971 the fundamental concept of strange attractor. It is an attractive geometric object, different from the previously known cases of periodic fixed points, quasi-periodic fixed points or limit cycles, hence the name "strange", and which also has a non-integer or fractional (fractal) dimension.

On the other hand, the development of fractal geometry started by Benoit Mandelbrot [14], who had been a student of the French mathematician Gaston Julia, has played a fundamental role in the understanding and analysis of the complex behavior of nonlinear dynamical systems. In any case, it is important not to forget the role played in many aspects of the development of nonlinear dynamics by the German mathematician Georg Cantor (1845-1918),

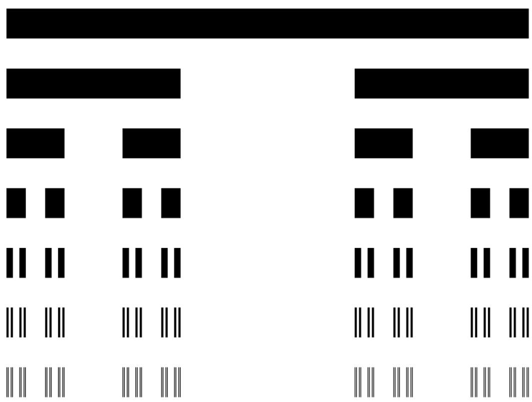

Fig. 15. The middle-third Cantor set is one of the simplest fractal sets

particularly with regard to the middle-third Cantor set (Fig. 15) and its constant appearance in many dynamic problems. This will be discussed in more detail in section 3.6.

3.3 Statistical Mechanics, Origin of Irreversibility and Ergodic Theory

Statistical mechanics is an essential part of theoretical physics whose purpose is to describe the macroscopic properties of a very large system of particles in terms of their averaged properties. It is a discipline that combines the basic laws of dynamics for a particle system along with the laws of statistics, especially those concerning the law of large numbers. The discovery of deterministic chaos has stimulated some physicists to reconsider from a new perspective the foundations of statistical mechanics. This is because deterministic chaos implies that not only systems with a large number of particles, but even deterministic systems with very few degrees of freedom can present behaviors that require statistical tools for their study. Many efforts have been made during this last century to give a correct interpretation of the dynamical origins of irreversibility. Despite all the efforts made to date, there is still no general agreement on what are the essential ingredients needed to support statistical mechanics.

Fig. 16. The founding fathers of Statistical Mechanics: Ludwig Boltzmann (1844-1906), James Clerk Maxwell (1831-1879 and Josiah W. Gibbs (1839-1903)

The problem of irreversibility was one of the major concerns of one of the "founding fathers" of statistical mechanics (Fig. 16), the Viennese physicist Ludwig Boltzmann (1844-1906). The objection raised by Josef Loschmidt (1821-1895) to Boltzmann's program, consisting in deriving the laws of thermodynamics directly from mechanical behavior, revealed the paradoxical of a situation in which, while the laws of mechanics are reversible under temporal inversion, the thermodynamic behavior of the systems is fundamentally irreversible. There has certainly been great progress in this century in the attempt to clarify the dynamic origin of the kinetic equations, although the problem remains to some extent open. Following Boltzmann, the first attempts to substantiate classical statistical mechanics were based on the supposed validity of the ergodic hypothesis, which, after making considerable theoretical efforts, led to a true impasse.

Following the work of Maxwell and Boltzmann, Gibbs introduced the concept of a "Mixing" associated to a system using the simile of an oil drop in an immiscible fluid, a small region in the phase space that simulates the oil drop, the dynamical evolution would help fill the entire phase space. This idea implies that for a given dynamical system, two sufficiently close points would separate exponentially after a certain period of time. This concept is linked to the notion of sensitive dependence on initial conditions that is at the base of chaotic dynamics in nonlinear dynamics and that leads to define the so-called Lyapunov exponents [15]. The concept of Lyapunov exponent indicates that if a dynamical system has any positive Lyapunov exponent then these initial points or conditions would separate exponentially and this type of systems are called chaotic systems, since the prediction of the evolution of the system in the long term is impossible.

In this sense, scientists like George Birkhoff (1884–1944) stand out, who proposed the ergodic theorem, which was later proved by the German mathematician Eberhard Hopf (1902-1983) using the fact of the ergodicity of the trajectories on surfaces of constant negative curvature that French mathematician Jacques Hadamard had pointed out a few years earlier. However, these results had little impact on the foundation of nonequilibrium statistical mechanics.

The importance of the Lorentz gas, which was previously mentioned when talking about Max Born, is that it shows thermodynamic physical properties, is ergodic and has a positive Lyapunov exponent. The great achievement of Russian American mathematician Yakov Sinai, who received the Abel Prize in 2014 (Fig. 17), was to show the connection between the classical Boltzmann-Gibbs set for an ideal gas and a chaotic Hadamard billiard.

Ideas from chaos theory have been used for the foundation of statistical mechanics, finding deep connections between the dynamical properties of a system, such as its Lyapunov exponents and its transport properties. Knowledge of both the nonequilibrium statistical mechanics and nonlinear dynamics is essential to understand works on nonequilibrium states. Despite numerous efforts and apparent new perspectives to support the nonequilibrium statistical mechanics based on chaos theory, the extraordinary conceptual difficulties of such an undertaking have so far prevented its achievement.

Fig. 17. The American-Russian mathematician Yakov G. Sinai with Crown Prince Haakon of Norway when he received the 2014 Abel Prize.

3.4 The Path towards Chaos through Nonlinear Oscillators

The construction of nonlinear dynamics is like that of a large river to which numerous tributaries contribute. One of these tributaries is the study of nonlinear oscillators. Among the pioneers in this path we can find the English physicist John William Strutt, Lord Rayleigh (1842-1919), motivated by his interest in understanding the physics of musical instruments. For this type of system, a first approximation based on the use of linear oscillators is not effective because the real instruments do not produce a simple tone, as it happens to a linear oscillator, so it is necessary to add friction on one side and terms not linear recovery by another. That is, it is necessary to use an elastic force different from that provided by Hooke's law: *ut tensio sic vis*. By clever use of the basic dynamical elements of the problem, Lord Rayleigh created models that explained the sound emitted by musical instruments. In his famous book *The Theory of Sound* published in 1877, Rayleigh introduced a series of fairly general methods such as the notion of a limit cycle, which is a periodic motion that has the physical system regardless of the initial conditions.

German engineer Georg Duffing (1861-1944) is known primarily for his symmetric nonlinear oscillator model with a cubic nonlinearity: Duffing oscillator. This model is a paradigmatic model for the study of many phenomena in nonlinear dynamics. The theory was later developed in the late 1940s, just after World War II, by two English mathematicians at Cambridge University: Mary L. Cartwright (1900-1998) and John E. Littlewood (1885-1977) who showed that many of the experiments of experimental physicists and many of the conjectures of theoretical physicists were derived directly from the analysis of differential equations of motion. In fact, these mathematicians had followed the ideas of George Birkhoff.

The school of nonlinear thought in Russia was started by the work of Leonid I. Mandelstam (1879-1944) on nonlinear oscillators, who had trained with the German physicist August Kundt (1839-1894) in Strasbourg, well known for his works on acoustics and the Kundt tube. This line of work was continued by Alexander A. Andronov (1901-1952) (Fig. 18) and by Lev S. Pontryagin (1908-1988), who introduced the notion of structural stability of a system of equations, a concept associated with that of bifurcations of dynamical systems.

Fig. 18. The Russian mathematician Alexander A. Andronov (1901-1952) one of the pioneers in nonlinear science.

The concept of bifurcation of limit cycles that had been suggested by Poincaré in 1892, was tested by Andronov in 1930 and by Hopf in1940, and is called the Andronov-Hopf bifurcation, although it is better known simply as a Hopf bifurcation. This school continued later in the 50s and 60s in Gorky, current Nizhnii Novgorod, obtaining parallel results to the development of the theory in the West. Many methods of nonlinear physics were developed under the paradigm of nonlinear oscillators and self-oscillations.

Another important school on nonlinear thought in Russia was the Kiev School of Nonlinear Oscillations Research that was initiated by Nikolai M. Krylov (1879–1955) and his student Nikolai N. Bogolyubov (1909–1992) (Fig. 19), at the beginnings of the 1930s. They developed much fundamental work on quasi-periodic solutions for non-autonomous systems and established the discipline of Nonlinear Mechanics as a part of Physics. Most of their work was published in the book *Introduction to Nonlinear Mechanics* (1937) [16] in Russian. An English version was published in 1943 by Princeton University Press after the translation carried out by the Russian mathematician Solomon Lefschetz (1884-1973) who led the Nonlinear Oscillation Project (ONR) in Princeton, and translated work available in Russian for the English-speaking world.

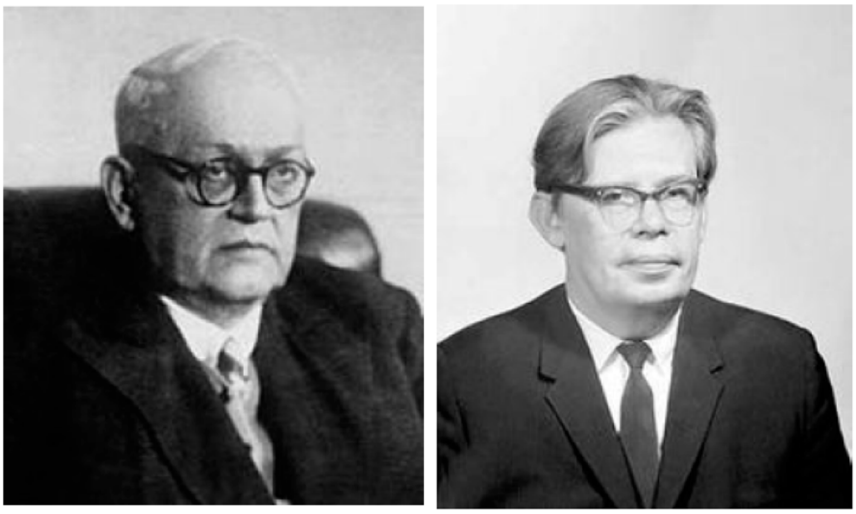

Fig. 19. Nikolai M. Krylov and his student Nikolai N. Bogolyubov who develop the Nonlinear Mechanics School in Kiev in the 1930s.

In Japan, the theory of nonlinear oscillators and their applications to radiophysics were developed at the school of Japanese engineer Chihiro Hayashi (1911-1986) at Kyoto University. Hayashi made notable contributions to the study of nonlinear oscillators and their practical applications in electrical engineering, publishing his famous book *Nonlinear Oscillations in Physical Systems* in 1964 [17].

In 1961 a remarkable event takes place on the part of the Japanese engineer Yoshisuke Ueda, who was a doctoral student of Chihiro Hayashi. Ueda studied the dynamics of various nonlinear oscillators such as the van der Pol oscillator and the Duffing oscillator, and it is precisely in a particular model of the latter that he apparently found solutions for the first time that we now designate as chaotic solutions.

3.5 Population Dynamics and the Logistic Map

The logistic map was popularized by Robert M. May (1936-2020) (Fig. 20) after the publication of his influential paper *"Simple mathematical models with very complicated dynamics"* [18], and constitutes one of the paradigms of the chaotic behavior of nonlinear dynamical systems. Robert May started his scientific career as a physicist, but soon he moved into biology becoming one of the pioneers in theoretical ecology, what led him to become a pioneer in chaos theory. Despite the apparent simplicity of the logistic map, it displays complex dynamics including chaotic behavior. Its formulation derives from the logistic equation, introduced in 1838 as a model of growth in population dynamics by the Belgian mathematician Pierre François Verhulst (1804-1849) in his writing *"Notice sur la loi que la population poursuit dans son accroissement"*. The quadratic map, very similar to the logistic map, had also been extensively studied in other contexts by the French Gaston Julia (1893-1978), by the Hungarian-American John von Neumann (1903-1957), and by the American Norbert Wiener (1894-1964).

Fig. 20. Robert M. May (1936-2020), Baron May of Oxford. A physicist and pioneer of theoretical ecology who led him to contribute to chaos theory.

One of the most influential articles in the field was undoubtedly the previously quoted article by Edward Lorenz *Deterministic nonperiodic flow* [13]. The American mathematician and physicist at the University of Maryland Prof. James A. Yorke (Fig. 21) immediately recognized the implications of such a discovery, as well as its philosophical repercussions, making Lorenz's work known to the scientific community.

Fig. 21. The American mathematician and physicist James Yorke. He coined the term chaos in the modern scientific literature.

Later he introduced the term *chaos* in the article entitled *Period Three Implies Chaos* [19] published together with his PhD student Tien-Yien Li in The American Mathematical Monthly magazine in 1975. A few years earlier, in 1963, the Ukrainian mathematician A. N. Sharkovskii had proved a theorem (now known as Sharkovskii's theorem), which was published in Russian in the Ukrainian Mathematics Journal, and where part of the Li and Yorke's result appeared as a corollary. However, one of the fundamental novelties in the article by Li and Yorke is that they wrote that the appearance of a period three orbit implied the appearance of all the others, including the chaotic orbits, while Sharkovskii did not talk about the chaotic orbits.

Subsequently, the American physicist Mitchell Feigenbaum (1944-2019) discovered the existence of universal critical exponents that characterized the transition from periodic to chaotic motion in one-dimensional maps with the property of period doubling. Simultaneously, the same discovery was made by the French Pierre Coullet and Charles Tresser, who at the time were doctoral students at the University of Nice, and by the German physicists at the University of Marburg, Siegfried Grossmann and Stefan Thomae.

The renormalization group concept had previously been applied in the field of statistical mechanics to study the so-called critical phenomena and phase transitions and its development in these fields had earned the Nobel Prize for the American physicist Kenneth Wilson in 1982. These methods were applied by Feigenbaum and others to develop the mathematical theory of period doubling bifurcations. Until the beginning of the eighties, most of the works were of a theoretical nature or the result of numerical explorations with computers. In any case, the important consequences that these theoretical discoveries had for physics were always considered, as well as the possible importance for understanding the transition to fluid turbulence.
French physicist Albert Libchaber (Fig. 22), currently at Rockefeller University in New York, carried out one of the first experiments where the phenomenon of period-doubling was shown when studying Rayleigh-Bénard convective cells in the late 1970s. American physicist Robert Shaw of the University of California at Santa Cruz performed a simple and particularly relevant experiment with a simple dripping faucet. Another important experimental milestone was carried out by the American physicists Jerry Gollub and Harry Swinney (Fig. 22), who also found the period doubling phenomenon by reproducing the classical Taylor-Couette experiment of fluid motion. Their contributions to the experimental verification of some of some of the ideas derived from chaos theory have stimulated much experimental work in nonlinear dynamics and chaos.

Fig. 22. The physicists Albert Libchaber, Harry Swinney and Jerry Gollub have been pioneers on experimental work on chaos.

3.6 Fractional dimensions, fractals and chaos

There are many complex geometric shapes in nature such as shorelines, river beds, the biological forms and even the complex curves of the financial markets. A common feature in all of them is self-similarity. This is the property that consists in that when a part of this form is increased, the same type of structure appears. To characterize objects with this universal property, the use of fractional dimensions is necessary, which led to the physicist and mathematician Benoit Mandelbrot (1924-2010) (Fig. 23) to call these objects "fractals". His work in collecting the enormous work that had been done by mathematicians before him such as the French Gaston Julia, the Swedish Helge von Koch, the Polish Wacław Sierpiński, as well as the works on dimensions by the German Felix Hausdorff and the Russian Abram S. Besikovich had a remarkable influence that he gave to the field of fractal geometry.

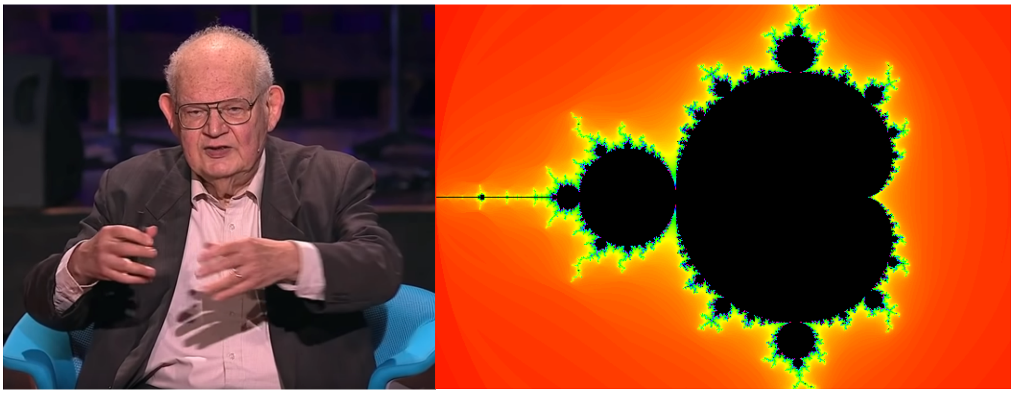

Fig. 23. Benoit Mandelbrot (1924-2010) and the famous set that bears his name.

The notion of dimension is fundamental when measuring geometric objects. There are several ways to define the concept of dimension, but it is clear that a point has dimension zero, a straight line has dimension one, a plane has dimension two and a cube has dimension three. However, and as strange as it may seem, there are geometric objects whose dimensions are not an integer, turning out to be a fractional value.

This is a simple notion of what is meant by a fractal dimension or a Hausdorff dimension, such that the Cantor set, mentioned above, has a dimension from $\log 2 / \log 3 \approx 0.63$, the Koch curve has a dimension of $\log 3 / \log 4 \approx 1.26$ and the Sierpinski set has a dimension of $\log 3 / \log 2 \approx 1.585$. All of them are self-similar fractal sets, since they are obtained by means of an iterative rule so that the basic structure is repeated at all scales.

The Koch curve (Fig. 24) was devised by the Swedish mathematician Helge von Koch (1870-1924), and is constructed as follows: We start with an interval that we divide into three equal pieces, and in the middle piece we build a triangle and equal sides as it appears in the figure.

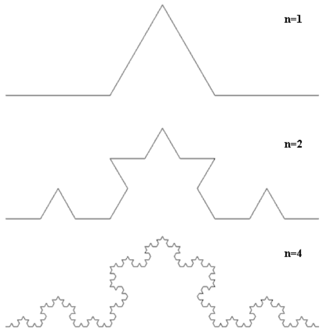

Fig. 24. The Koch curve is a fractal set

Next, we repeat the same strategy in each of the four pieces, giving rise to the figure in the middle, and we repeat the process in successive iterations giving place to a figure that resembles a snowflake. The Sierpinski fractal is due to the Polish mathematician Wacław Sierpiński (1882-1969) and is constructed as follows. We consider a triangle with equal sides, like the one shown in Fig. 25. Next, we remove the white triangle from inside it and in each of the remaining triangles we remove the white triangle and so on, finally giving rise to successive iterations to the Sierpinski triangle, which is a self-similar fractal object.

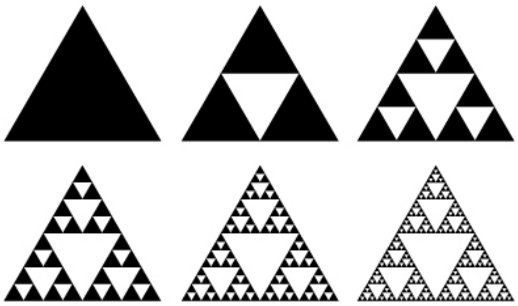

Fig. 25. Sierpinski fractal set

Although in principle fractal geometry and nonlinear dynamics are two disciplines that apparently have nothing to do with it, nevertheless, as previously noted, chaos and fractals are intimately linked. One of the main ideas is due to the fact that associated with the notion of chaos exists that of the chaotic attractor that constitutes a geometric object of a fractal nature that lives in the phase space, so that it is impossible to speak of chaos without speaking of fractals and vice versa.

4. On the Origins of Complexity

At the beginning of the 20th century, fundamental developments took place in two new fields of research in Physics that represent a huge conceptual revolution in the development of science. On the one hand, the theory of relativity that helped us understand the world on cosmic scales and quantum mechanics that involved the knowledge and exploration of the microscopic world at the atomic and subatomic levels. On the other hand, during the second half of the 20th century, we have been able to see how nonlinear dynamics and chaos theory emerged as one of the very fruitful fields of activity in research. Likewise, the discipline of complexity, or the physics of complex systems, has received a huge push, including new lines of research and bringing a new way of doing things.

Fig. 26. Warren McCulloch (1898-1969)

Talking about the origins of things is never easy and of course the origins of complexity are no exception. In spite that for many it is a relatively new notion, since its use has become widespread in recent years, its origins date back to much earlier times. When exploring certain ideas and activities that have contributed to the development of this set of ideas that complexity encompasses, it is worth mentioning the American neuroscientist Warren McCulloch (Fig. 26), who together with the mathematician Walter Pitts, proposed in 1943 the well-known McCulloch-Pitts neuron model to analyze brain properties. McCulloch also played a prominent role in the organization in the 1940s of the Macy Conferences, with the support of the Macy Foundation, where numerous scientists from various disciplines participated in a highly interdisciplinary environment, among which we can mention the psychiatrist William Ross Ashby, the anthropologist Gregory Bateson;

mathematicians John von Neumann, Walter Pitts and Norbert Wiener, biophysicist Max Delbrück, information theorist Claude Shannon and Warren McCulloch himself as moderator.

On the other hand, it is of special interest the figure of the American scientist Warren Weaver (1894-1978) (Fig. 27), who among other things was co-author with Claude E. Shannon of the famous book *The Mathematical Theory of Communication* published by The University of Illinois Press in 1949.

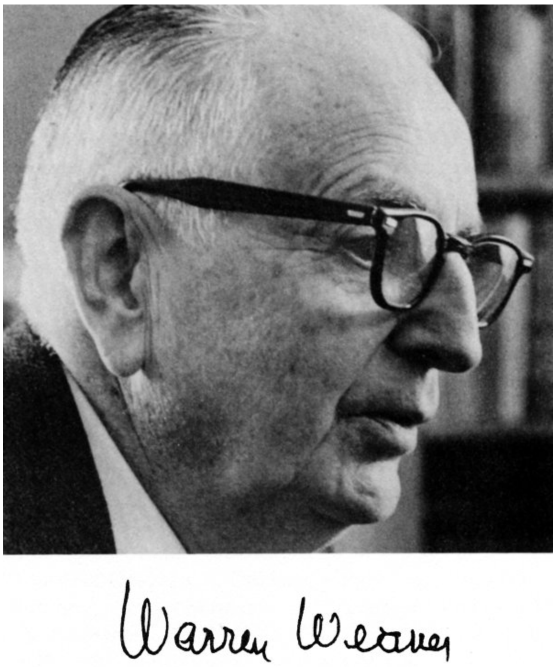

Fig. 27. Warren Weaver (1894-1978) pioneer in the use of computers in scientific research.

In 1948 he published a very interesting article, considered foundational, entitled *Science and Complexity* [20] in the American magazine American Scientist. In fact, he used material that had been published in 1947 and the most important thing is that it is premonitory of many aspects of the complexity that have been discussed in recent years.

4.1 Physics and Emergence

One of the fundamental ideas in complexity is the idea of emergence. In physics there are numerous examples of systems where emerging properties are evident, such as superconductivity and superfluidity. It should also be noted that there is all a fundamental research that seeks to investigate complex phenomena, where instead of resorting to reductionism, which has been the approach that has governed the evolution of Physics in recent years, the primary engine of this research is the emergence. A fundamental point is that these emerging complex phenomena do not derive from the underlying microscopic laws, although of course they do.

Some of these ideas were masterfully presented by the physicist Philip W. Anderson (1923-2020) (Fig. 28), 1977 Nobel Prize in Physics, in an article published in the journal Science in 1972 and entitled *More is different* [21], where he leaves very clear the idea that:

"At each level of complexity entirely new properties appear, and the understanding of the new behaviors requires research which I think is as fundamental in its nature as any other."

Fig. 28. Philip W. Anderson (1923-2020), 1977 Nobel Prize in Physics.

Philip W. Anderson introduces some aspects of the physics of complex systems in the article entitled Physics: The Opening to Complexity [22], where he points out, among other things:

"But another large fraction are engaged in an entirely different type of fundamental research: research into phenomena that are too complex to be analyzed straightforwardly by simple application of the fundamental laws. These physicists are working at another frontier between the mysterious and the understood: the frontier of complexity. At this frontier, the watchword is not reductionism but emergence. Emergent complex phenomena are by no means in violation of the microscopic laws, but they do not appear as logically consequent on these laws."

In relation to the physics of emergence, it is also worth mentioning Robert Laughlin, 1998 Nobel Prize in Physics and professor at Stanford University, who proposed to his best students the problem of deducing the laws of superfluidity from first principles, knowingly that it is impossible. Precisely to show them the importance of emergent properties in physics, which is the fundamental argument of his book *A Different Universe: Reinventing physics from the bottom down* [23].

The book is based on an interesting article entitled *The Science of Everything* [24], where among the many questions he points out we can highlight the following two paragraphs:

The central task of theoretical physics in our time is no longer to write down the ultimate equations but rather to catalogue and understand emergent behavior in its many guises, including potentially life itself. We call this physics of the next century the study of complex adaptive matter. For better or worse we are now witnessing a transition from the science of the past, so intimately linked to reductionism, to the study of complex adaptive matter, firmly based in experiment, with its hope for providing a jumping-off point for new discoveries, new concepts, and new wisdom.

"End of Reductionism, for it is actually a call to those of us concerned with the health of physical science to face the truth that in most respects the reductionist ideal has reached its limits as a guiding principle. Rather than a Theory of Everything we appear to face a hierarchy of Theories of Things, each emerging from its parent and evolving into its children as the energy scale is lowered. The end of reductionism is, however, not the end of science, or even the end of theoretical physics."

In fact, when one looks at the world what one observes is of amazing complexity. Although, for the moment, there are no laws of complexity, as there are laws of physics, the authors cited above list a number of simple lessons on complexity that derive from the analysis and observation of numerous complex systems that exist in the universe.

Hungarian physicist Tamas Vicsek from the Department of Biophysics at Eötvös University in Budapest argues in an essay published in Nature [25] that when a concept is not well defined, as is the case with complexity, there is a danger of abusing it. It is true that on many occasions the term can be used indiscriminately as a sign of modernity. However, the fundamental idea derived from this essay is that the laws that describe the behavior of complex systems are qualitatively different from those that govern the units of which they are composed.

4.2 Complexity and Life Sciences

The enormous development of scientific activity in recent years has caused many disciplines to find fields of application in other sciences. This is what, among many other cases, has happened with the application of disciplines such as physics, mathematics and engineering in the development of some aspects of the life sciences, in which we could include not only biology, but also biomedical sciences and biotechnology. You might think that it is a simple fashion and something that for some reason has been happening for just a few years. However, it is important to note that the influence of these sciences and their contributions to the life sciences are very old.

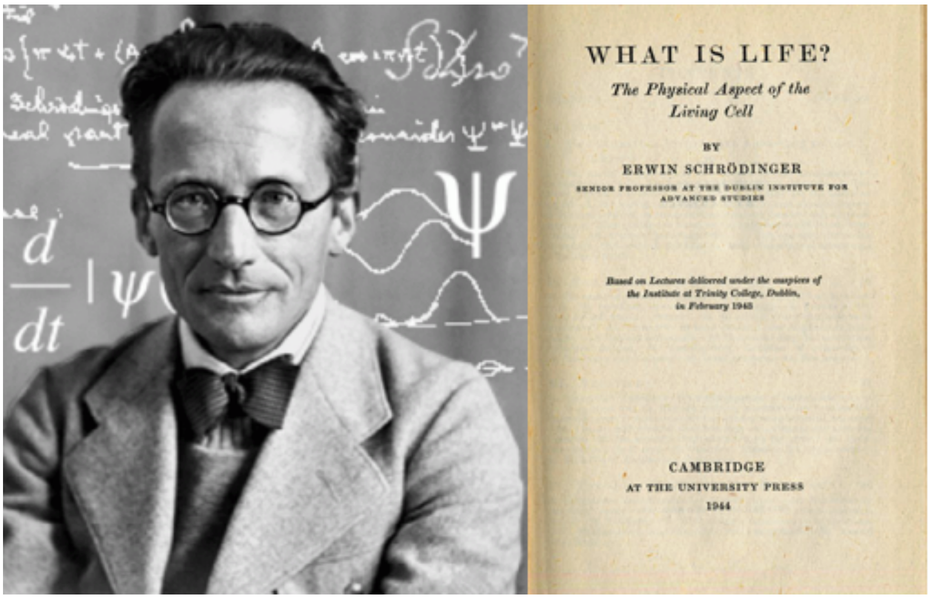

Fig. 29. Erwin Schrödinger, 1933 Nobel Prize in Physics and known above all for his contributions to quantum mechanics, who authored the influential book *What is life?* published in 1944

There are many eminent physicists, including Nobel Laureates, whose work has been related in some of the aspects related to the issues of complexity. Among them are: Erwin Schrödinger, 1933 Nobel Prize in Physics and known above all for his contributions to quantum mechanics, who authored the influential book *What is life?* published in 1944 (Fig. 29). Physicist Max Delbrück (1906-1981), Nobel Prize in Medicine in 1969 for his pioneering work in Molecular Biology. Philip W. Anderson, 1977 Nobel Prize in Physics, well known for his work in condensed matter physics, has also played a relevant role in the development of some ideas related to complexity, especially emergence. Physicist Murray Gell-Man, 1969 Nobel Prize in Physics; who coined the term *quark*.

Another fundamental character in this relationship that we are making is the mathematician Norbert Wiener (1894-1964) (Fig. 30), professor at the Massachusetts Institute of Technology (MIT), who was one of the founders of Cybernetics, and knew how to create a highly interdisciplinary environment around him with numerous applications to life sciences. We could continue quoting numerous physicists, such as Nicholas Metropolis, George Gamow, Leo Szilard, Jack Cowan or Geoffrey West.

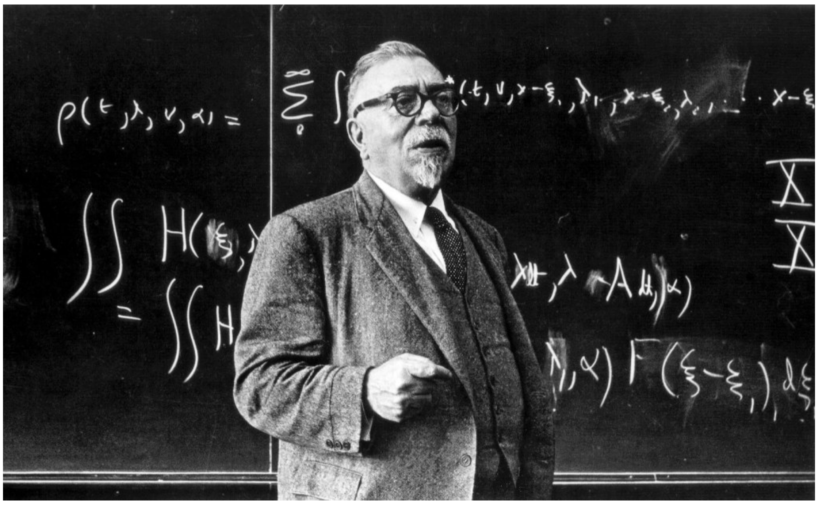

Fig. 30. Norbert Wiener (1894-1964) one of the founders of Cybernetics.

Among the most widely used mathematical models in computational neuroscience, which aim to analyze the brain as a complex system, we can consider the Hodgkin-Huxley model. In 1952 Alan L. Hodgkin and Andrew. F. Huxley (Fig. 31) wrote a series of five articles [26] in which they described the experiments they carried out to determine the laws of ion motion in nerve cells during an action potential.

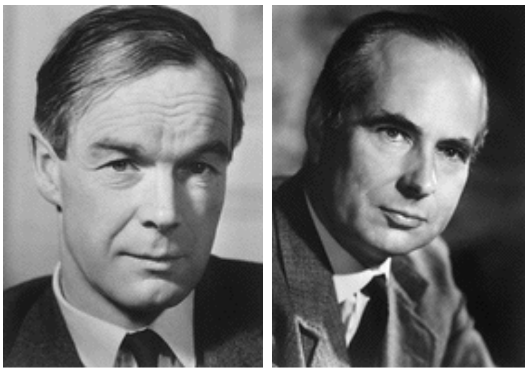

Fig. 31. Alan L. Hodgkin and Andrew F. Huxley received the Nobel Prize in Nobel Prize in Physiology or Medicine in 1963 for his neural model.

They formulated a mathematical model to explain the behavior of nerve cells in a giant squid. Remarkably, this model was formulated long before the existence of electron microscopes and computer simulations and allowed scientists basic knowledge of how nerve cells function without knowing how membranes behaved. They received the Nobel Prize in Physiology or Medicine in 1963, along with Sir John C. Eccles for their discoveries regarding the ionic mechanisms involved in excitation and inhibition in the peripheral and central portions of the nerve cell membrane.

5. Conclusions

One of the key ideas that should be highlighted here is that although the physics of complex systems is currently one of the frontiers of current physical research, the ideas of complexity go back to the beginning of the 20th century and have been developed along various paths until we reach the vision we have as of today, although their evolution and development throughout the 21st century are quite open.

The concept of emergence versus reductionism is another of the fundamental ideas in the physics of complex systems. The ones concerning emergence go back even to the origins of thermodynamics, and they appear in various phenomena studied by that science. Concepts such as chaos and fractals are mentioned in a special way, which have been a catalyst for many of the notions around which complexity moves. Without a doubt, interdisciplinarity is of the utmost importance in this context, since, as it has been pointed out, many ideas associated with complexity help to integrate disciplines, as well as breaking traditional disciplinary barriers.

At all times it has been wanted to put on record that many of the ideas discussed in this article have been beating in the thought and action of many physicists in the past and present, who have been open to problems about the complexity of life and nature, including some Nobel Prizes.

In recent years, numerous scientists have contributed to the development of chaos theory. In 2003 the Japan Prize, which is awarded each year by the Japanese government through the *The Japan Prize Foundation* was dedicated to Complexity Science and Technology. The award was won by the scientists Benoit Mandelbrot for his contributions to fractals and James A. Yorke for his contributions to the foundation of chaos theory. This award was very special to the community of scientists working in these fields, since for the first time an award of this magnitude was awarded to scientists working on complex science issues.

Following the efforts of numerous scientists, as we have just shown, the entire field of research covering nonlinear dynamics, chaos theory and complexity continues to develop and influence numerous disciplines with new methods and novel ideas, showing great prospects for the future.

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