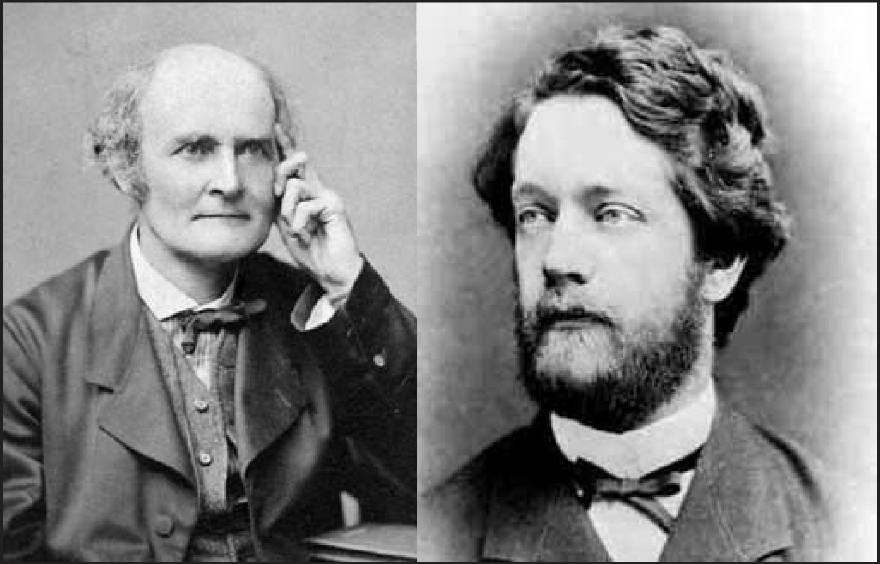
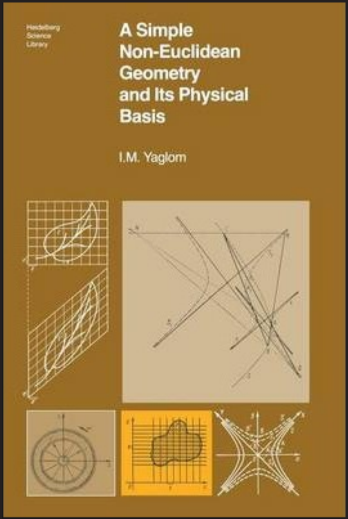
Universidad Rey Juan Carlos, Diciembre 2019

Jornada Homenaje a Miguel Ángel Fernández Sanjuán en sus 60



Cayley-Klein geometries: a modern historical perspective

Mariano Santander Universidad de Valladolid



- Estudios de Licenciatura
  - ⋆ Universidad de Valladolid
- Tesina:
  - ⋆ Contracciones ultrarelativistas del grupo de Poincaré, 1981
- I. M. Yaglom

The Nine Geometries of Cayley-Klein type

- **The nine two-dimensional CK spaces $S^2_{[\kappa_1],\kappa_2} = SO_{\kappa_1,\kappa_2}(3)/SO_{\kappa_2}(2)$.**
  - ✓ Singularized by two real parameters κ_1, κ_2
- ⋆ **Values $\kappa_1 > 0, = 0, < 0$ denoted $+, 0, -$ label columns; similarly κ_2 vs. rows.**

Spherical: $\mathbf{S}^2$ $S^2_{[+],+} = SO(3)/SO(2)$	Euclidean: $\mathbf{E}^2$ $S^2_{[0],+} = ISO(2)/SO(2)$	Hyperbolic: $\mathbf{H}^2$ $S^2_{[-],+} = SO(2,1)/SO(2)$
Oscillating NH: $\mathbf{NH}^{1+1}_{+}$ (Co-Euclidean) $S^2_{[+],0} = ISO(2)/ISO(1)$	Galilean: $\mathbf{G}^{1+1}$ $S^2_{[0],0} = IISO(1)/ISO(1)$	Expanding NH: $\mathbf{NH}^{1+1}_{-}$ (Co-Minkowskian) $S^2_{[-],0} = ISO(1,1)/ISO(1)$
Anti-de Sitter: $\mathbf{AdS}^{1+1}$ (Co-Hyperbolic) $S^2_{[+],-} = SO(2,1)/SO(1,1)$	Minkowskian: $\mathbf{M}^{1+1}$ $S^2_{[0],-} = ISO(1,1)/SO(1,1)$	De Sitter: $\mathbf{dS}^{1+1}$ (Doubly Hyperbolic) $S^2_{[-],-} = SO(2,1)/SO(1,1)$

- **All these geometries appear realized in Nature**

The many faces of κ_1, κ_2 [1]

- **Projective view**
  - ⋆ $\kappa_1 >, =, < 0$ ≡ Eliptic, Parabolic, Hyperbolic type of measure of **distances**
  - ⋆ $\kappa_2 >, =, < 0$ ≡ Eliptic, Parabolic, Hyperbolic type of measure of **angles**
- **Old synthetic view**
  - ⋆ $\kappa_1 >, =, < 0$ ≡ a number $0, 1, \infty$ of lines through a given point P and not meeting a given line l (not through P).
  - ⋆ $\kappa_2 >, =, < 0$ ≡ a number $0, 1, \infty$ of points on a given actual line l and not joined to a given point P by an actual line $(P$ not on $l)$.
- **Differential Geometry view**
  - ⋆ $\kappa_1 >, =, < 0$ ≡ Positive, Zero, Negative **constant curvature** κ_1
  - ⋆ $\kappa_2 >, =, < 0$ ≡ Positive Definite, Degenerate, Lorentzian, metric reducible to $diag\{+1, \kappa_2\}$ at each point: **signature** κ_2.
- **Limiting view:** Cases where either κ_1, κ_2 are zero are limiting approximations to the generic cases where both κ_1, κ_2 are different from zero
  - ⋆ $\kappa_1 \to 0$ limit around a point
  - ⋆ $\kappa_2 \to 0$ limit around a line

¿Two separate families? Riemannian . . . and Lorentzian

- **Riemannian spaces** Riemann's far-reaching extension of the Euclidean space $\mathbf{E}^n$
  - ⋆ **Two steps** Zero Curvature → Constant Curvature → General Curvature.
- **Constant curvature Riemannian spaces**
  - ✓ Essentially, a one-parameter family of n-d Riemannian spaces of constant curvature grouped in three types $\mathbf{S}^n_\kappa, \mathbf{E}^n, \mathbf{H}^n_k$ according as $\kappa > 0, = 0, < 0$.
  - ✓ Standard choice $\kappa = 1, 0, -1$ gives the three standard $\mathbf{S}^n, \mathbf{E}^n, \mathbf{H}^n$.
- **Lorentzian spaces** A (similar) extension of the Lorentz-Minkowski space $\mathbf{M}^{1,n}$
  - ⋆ **No essential changes from Riemannian**
- **Constant Curvature PseudoRiemannian (Lorentzian) spaces**
  - ✓ Essentially, a one-parameter family of (1+n)-d Lorentzian spaces of constant curvature $\mathbf{AdS}^{1+n}_\kappa$, $\mathbf{M}^{1+n}, \mathbf{dS}^{1+n}_k$ according as $\kappa > 0, = 0, < 0$.
  - ✓ Standard choice $\kappa = 1, 0, -1$ gives the three standard $\mathbf{AdS}^{1+n}_\kappa, \mathbf{M}^{1+n}, \mathbf{dS}^{1+n}_k$.

CK Spaces Versus CK Geometries

- **The spaces discussed so far can be seen under the Riemannian and Kleinian perspectives**
- **Simplest example: Ordinary Euclidean space E^2** is a symmetric homogeneous space of the Euclidean group $ISO(2)$, $\mathbf{E}^2 \approx ISO(2)/SO(2)$.
  - ✓ Elements of this space are the points in Euclidean geometry, and the involution Π_1 correspond to reflection in a point.
- **Yet there is another symmetric homogeneous space in Euclidean Geometry: The set of all lines in E^2** . This is $\tilde{\mathbf{E}}^{\mathbf{2}} \approx ISO(2)/ISO(1)$.
  - ✓ Elements of this space are the **lines** in Euclidean geometry, and the involution Π_2 correspond to reflection in a line.
- **(Symmetric) Geometry:** An interlinked set of homogeneous spaces associated to the same group G but with a set of **commuting** involutive automorphisms.

Symmetric homogeneous spaces of 'Cayley-Klein type' [1]

- **I discuss only the real $2d$ case,** everything works for the real, Hermitian complex and quaternionic spaces, in any n.
- **Look for $3d$ Lie groups G which allow for two commuting involutive automorphisms in the corresponding Lie algebra.**
  - ⋆ **These would provide two symmetric homogeneous spaces of the Lie group G**
- **Approach** Look in the common eigenbasis $\{P_1, P_2, J\}$ The more general such (quasi-simple) Lie algebra having $\Pi_{(1)}, \Pi_{(2)}$ as automorphisms depends on two real parameters κ_1, κ_2

$$\Pi_{(1)} : (P_1, P_2, J) \to (-P_1, -P_2, J), \quad \Pi_{(2)} : (P_1, P_2, J) \to (P_1, -P_2, -J)$$

$$[P_1, P_2] = \kappa_1 J \qquad [J, P_1] = P_2 \qquad [J, P_2] = -\kappa_2 P_1$$

- **Denote $SO_{\kappa_1,\kappa_2}(3)$ the Lie groups obtained by exponentiation**
  - ⋆ **One-parameter subgroup invariant under involution $\Pi_{(1)}$** generated by J:

$$\exp(\gamma J) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{\kappa_2}(\gamma) & -\kappa_2 S_{\kappa_2}(\gamma) \\ 0 & S_{\kappa_2}(\gamma) & C_{\kappa_2}(\gamma) \end{pmatrix} \qquad SO_{\kappa_2}(2)$$

Symmetric homogeneous spaces of Cayley-Klein type' [2]

- **Labeled Trigonometric functions:** Labelled 'cosine' $C_\kappa(x)$ and 'sine' $S_\kappa(x)$:

$$C_\kappa(x) := \begin{cases} \cos\sqrt{\kappa}\,x \\ 1 \\ \cosh\sqrt{-\kappa}\,x \end{cases} \qquad S_\kappa(x) := \begin{cases} \frac{1}{\sqrt{\kappa}}\sin\sqrt{\kappa}\,x & \kappa > 0 \\ x & \kappa = 0 \\ \frac{1}{\sqrt{-\kappa}}\sinh\sqrt{-\kappa}\,x & \kappa < 0 \end{cases} .$$

  ⋆ **Deformations of the two basic functions 1 and x**

- **Natural realization of the CK group $SO_{\kappa_1,\kappa_2}(3)$** as a group of linear transformations in an ambient linear space $R^3 = (x^0, x^1, x^2)$.

  ⋆ **Therefore $SO_{\kappa_1,\kappa_2}(3)$ acts in R^3 as linear isometries of a bilinear form** with $\Lambda_{\kappa_1,\kappa_2} = \text{diag}\{+1, \kappa_1, \kappa_1\kappa_2\}$ as metric matrix.

  ⋆ **CK space as Homogeneous symmetric space:** $S^2_{[\kappa_1],\kappa_2} \equiv SO_{\kappa_1,\kappa_2}(3)/SO_{\kappa_2}(2)$

- **Natural structures in these homogeneous symmetric spaces:**

  ⋆ **A canonical connection (compatible with the metric).**

  ⋆ **A metric coming from the Killing-Cartan form in the Lie algebra.** The metric is of **constant curvature** κ_1 and of **'signature type'** κ_2 .

- **Hence this family includes precisely** the spaces of constant curvature (either $> 0, -0, < 0$) and (quadratic) metric of either signature type

The nine CK 2d spaces as 'spheres' in ambient space coordinates

Spherical: $\mathbf{S}^2$ $S^2_{[+],+} = SO(3)/SO(2)$	Euclidean: $\mathbf{E}^2$ $S^2_{[0],+} = ISO(2)/SO(2)$	Hyperbolic: $\mathbf{H}^2$ $S^2_{[-],+} = SO(2,1)/SO(2)$
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⋆ **Weierstrass ambient description as 'CK spheres'**

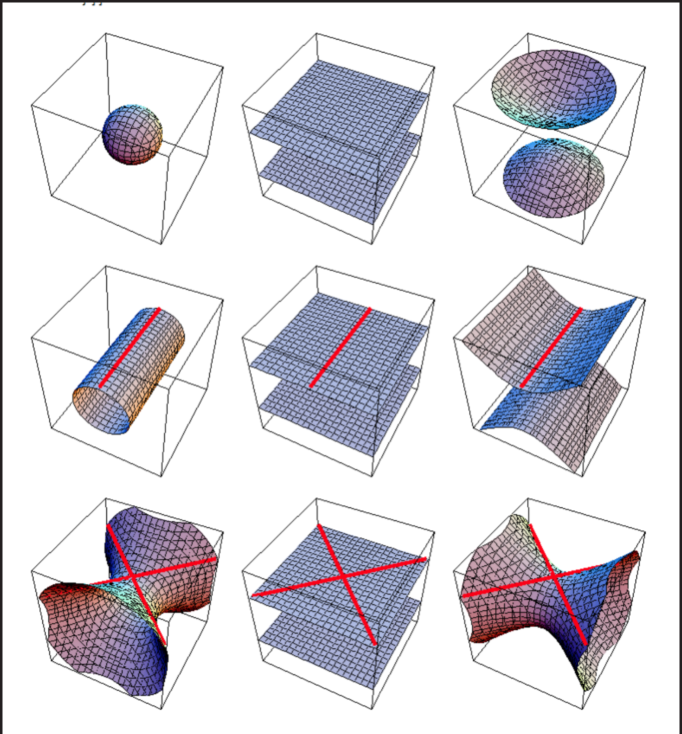
$$(x^0)^2 + \kappa_1 (x^1)^2 + \kappa_1\kappa_2 (x^2)^2 = 1$$

$\kappa_1 = 1, 0, -1$ (columns, left to right) $\kappa_2 = 1, 0, -1$ (rows, up to down)

⋆ **Metric in the ambient space** $dl^2 = (dx^0)^2 + \kappa_1 (dx^1)^2 + \kappa_1\kappa_2 (dx^2)^2$.

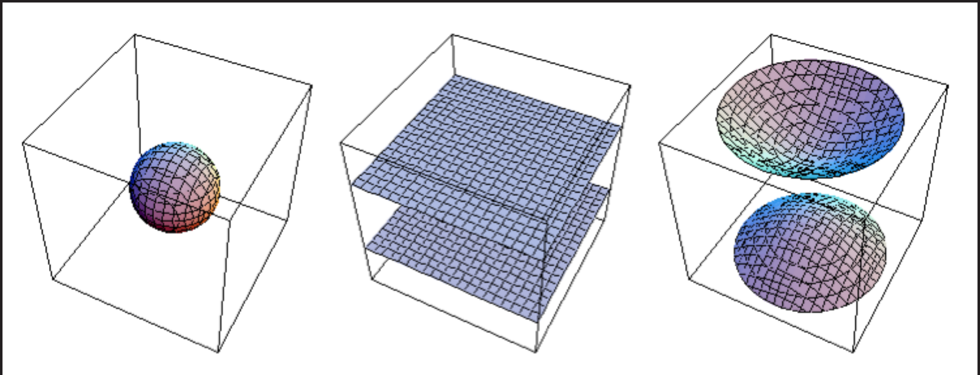
✓ Metric in the CK space $ds^2 := \frac{1}{\kappa_1} dl^2$

The nine CK 2d spaces as ‘spheres’ in ambient space coordinates



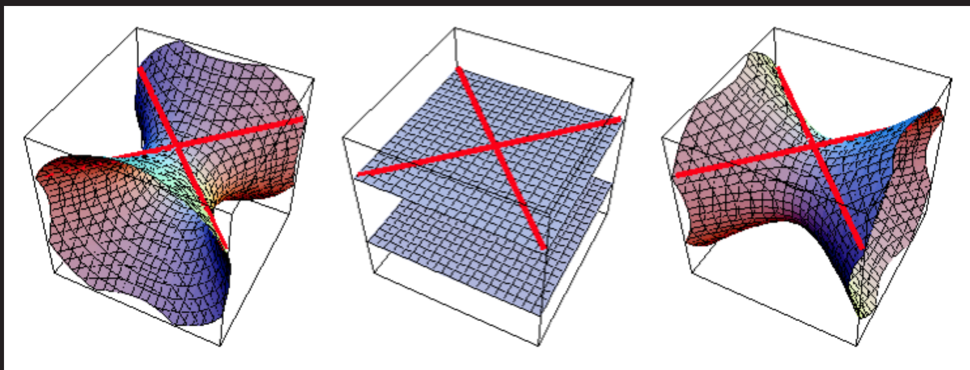
The three S^2, E^2, H^2 CK 2d spaces

✓ For distances r and angles θ, business as usual



Something new in the three AdS^2, M^2, dS^2 spaces? I

- ✓ In Minkowski space, rapidities appear through hyperbolic trig functions. $\kappa_2 < 0$ is a negative (hyperbolic) label. Standard choice $\kappa_2 = -1$
- ✓ But real rapidities ('angles') do not cover the full Minkowski, AntiDeSitter or DeSitter spaces
- ✓ Introduce a 'quadrant' with negative label $\kappa = -1$ defined so that $\llcorner\!\cdot_{-1}$ is the 'rapidity' between ortoghonal vectors in this space. $\llcorner\!\cdot_{-1} = \dfrac{\pi}{2\mathrm{i}}$
- ✓ Allow values $\chi,\ \chi + \llcorner\!\cdot_{-1},\ \chi + 2\llcorner\!\cdot_{-1},\ \chi + 3\llcorner\!\cdot_{-1}$ for the rapidity

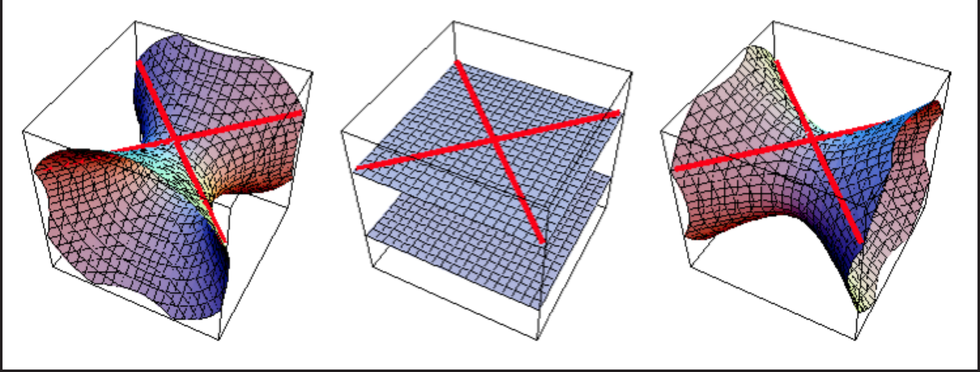


- ✓ Now rapidities cover the full Minkowski space (and AdS and dS too)

Something new in the three AdS^2, M^2, dS^2 spaces? II

⋆ **In Minkowski space ($\kappa_1 = 0$, proper times (the 'distances', denoted r) appear through parabolic trig functions**

- ✓ These real 'distances' do not cover the full Minkowski M^2
- ✓ The basic metric is a quadratic form which is not definite positive
- ✓ Introduce 'ideal' distances and allow the 'distances' to be either real r or pure imaginary $\mathrm{i}r$



- ✓ Now these 'distances' jointly with the extended 'angles' cover all M^2
- ✓ How about to extend this idea to all CK spaces?

Natural coordinates: beyond the reals . . .

- **Labeled Trigonometric functions:** Recall 'cosine' $C_\kappa(x)$ and 'sine' $S_\kappa(x)$ with 'label' κ are defined initially for real variable x (later for some particular complex values of x) as:

$$C_\kappa(x) := \begin{cases} \cos\sqrt{\kappa}\,x \\ 1 \\ \cosh\sqrt{-\kappa}\,x \end{cases} \qquad S_\kappa(x) := \begin{cases} \frac{1}{\sqrt{\kappa}}\sin\sqrt{\kappa}\,x & \kappa > 0 \\ x & \kappa = 0 \\ \frac{1}{\sqrt{-\kappa}}\sinh\sqrt{-\kappa}\,x & \kappa < 0 \end{cases}.$$

- **Define a quadrant:** $\llcorner\!\cdot_\kappa = \frac{\pi}{2\sqrt{\kappa}}$.

  ✓ Expression for the ambient space coordinates in terms of 'naive' polar coordinates

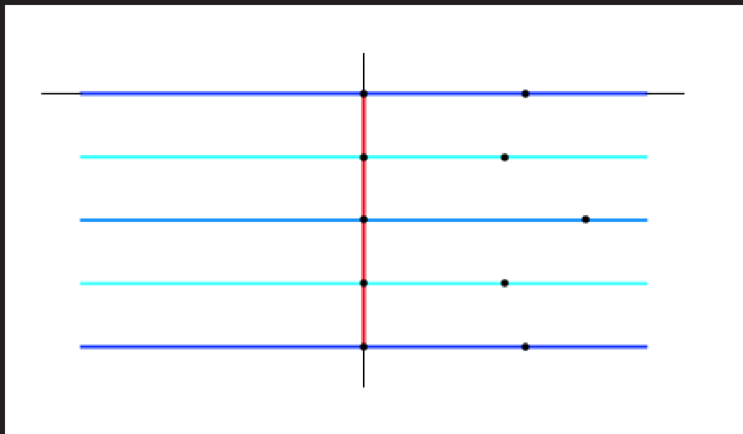
$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} C_{\kappa_1}(r) \\ S_{\kappa_1}(r)\,C_{\kappa_2}(\theta) \\ S_{\kappa_1}(r)\,S_{\kappa_2}(\theta) \end{pmatrix}$$

  ✓ but, what is the domain of the coordinates r, θ?

  ✓ For an hyperbolic quantity, (e.g, the minkowskian rapidity angle θ), $\llcorner\!\cdot_{-1}$ is a pure imaginary quantity. Should this mean that we have to accept any complex argument in the sine and cosine functions?

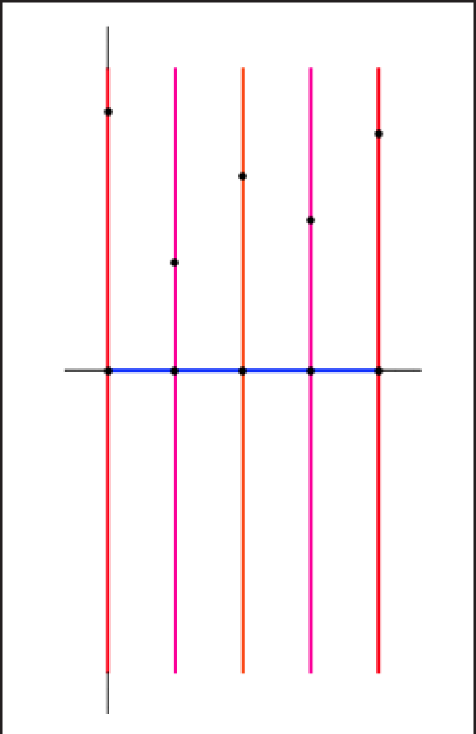
The full CK domain for the CK trigonometric functions

- ✓ No!. The natural requirement is to enforce that the squares of $S_\kappa(x)$ and $C_\kappa(x)$ should be real.
- ✓ This determines a subset of the complex plane, which is a kind of ‘branched one dimensional set’. This is called the full domain of the CK variable with label κ



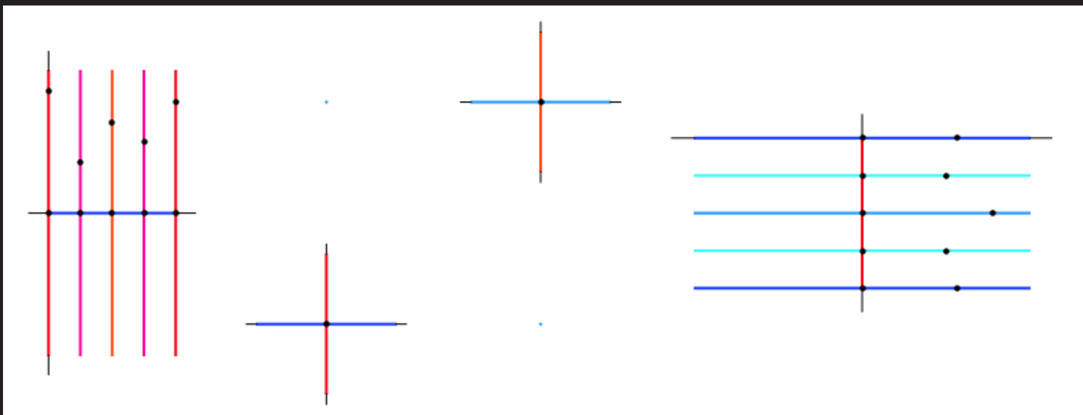
- ✓ This the the full CK domain with label $\kappa = -1$
- ✓ Branching points at $0,\ \pm\llcorner\cdot_{-1},\ \pm 2\llcorner\cdot_{-1} \equiv \theta_{-1}$ and as $\pm\infty$

The full CK domain of a variable with positive label κ



The full CK domain of a variable with any label κ

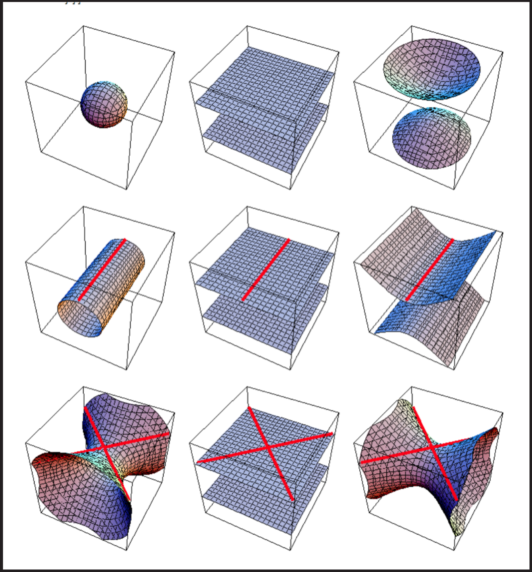
- ✓ The CK domain of a CK variable, for any value of κ is the set of the following values (here x, y are real, and $\theta \equiv 2\llcorner$)
- ✓ Actual and antiactual values, $x, \; 2\llcorner + x$ (depicted in deep blue)
- ✓ Coactual and anticoactual values, $\pm\llcorner + x$ (depicted in cyan)
- ✓ Ideal and antiideal values, $\mathrm{iy}, \; 2\llcorner + \mathrm{iy}$ (depicted in red and orange)
- ✓ Coideal and anticoideal values, $\pm\llcorner + \mathrm{iy}$ (depicted in magenta)

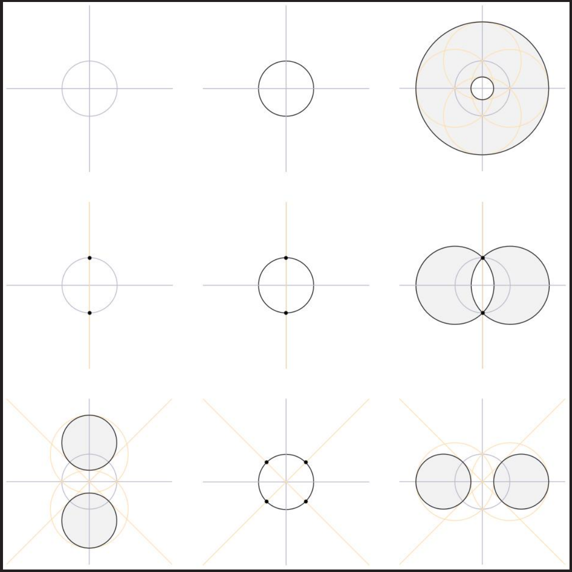


- ✓ Essential fact: The domain of a variable depends on its label

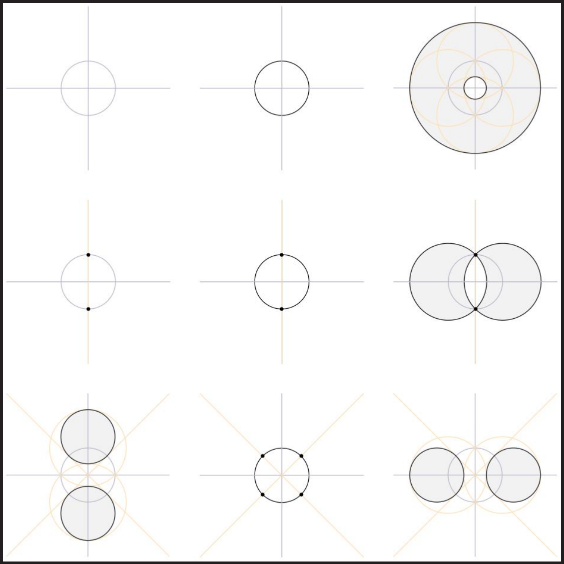
The stereocentral (stereognomonic) model of the nine CK spaces

- **A new 'projection' to display the nine spaces at once**
  - ✓ Essentially, extends the visually 'good' traits of the stereographic projection in the S^2





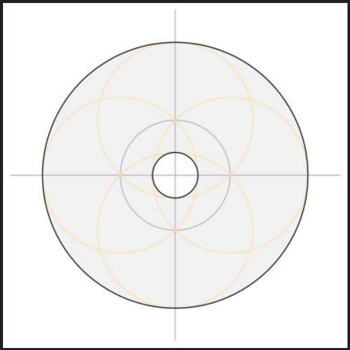
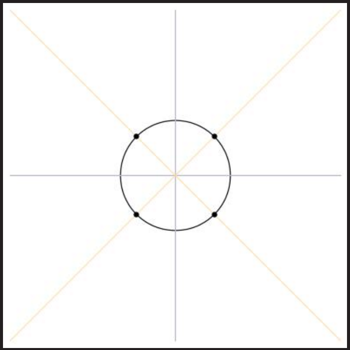
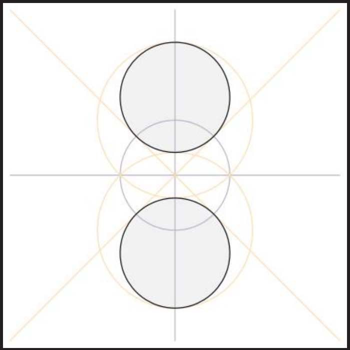
The stereocentral model of the nine CK spaces



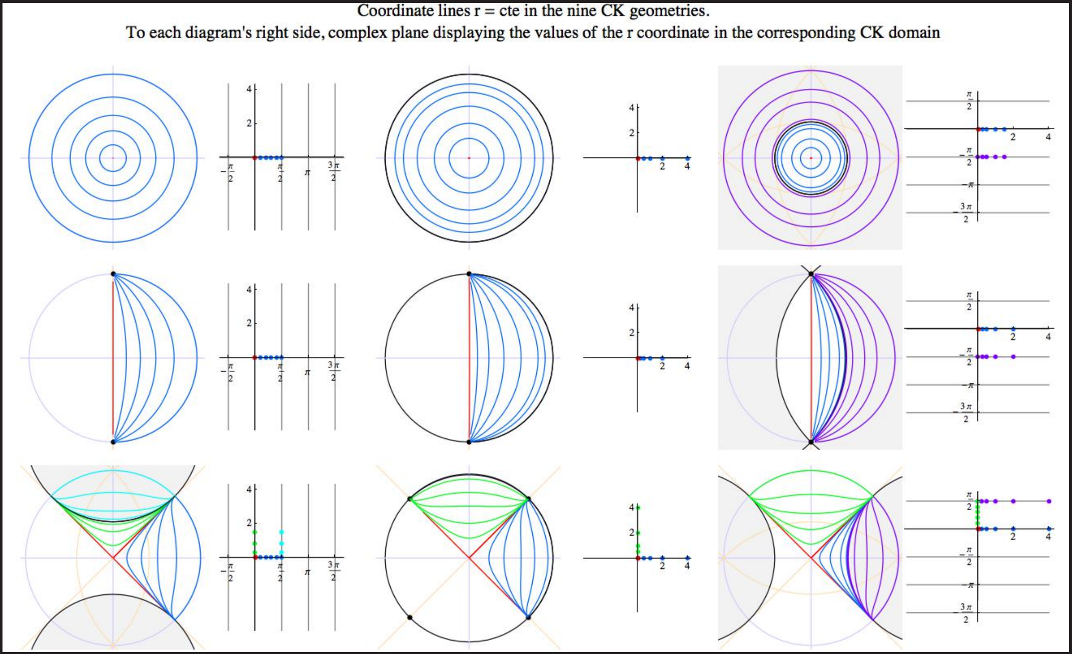
- ✓ For any CK space, geodesics are represented as ‘affine’ circles cutting the equator antipodally
- ✓ The ‘North’ and ‘South’ hemispaces are represented as the interior and exterior of the ‘Equator’ circle

The basic coordinate: distance to a point

- **Fix the point O at the origin**
  - ⋆ **For any other point P (in the 'full' CHK space) passes a unique geodesic linking P to O**
    - ✓ Antipodal exception
  - ⋆ **Define r the coordinate r as the 'extended' parameter of separation along this geodesic**
- **This r is defined in the 'full' CK space**
    - ✓ r could be either actual, coactual or ideal, coideal, or its anti versions

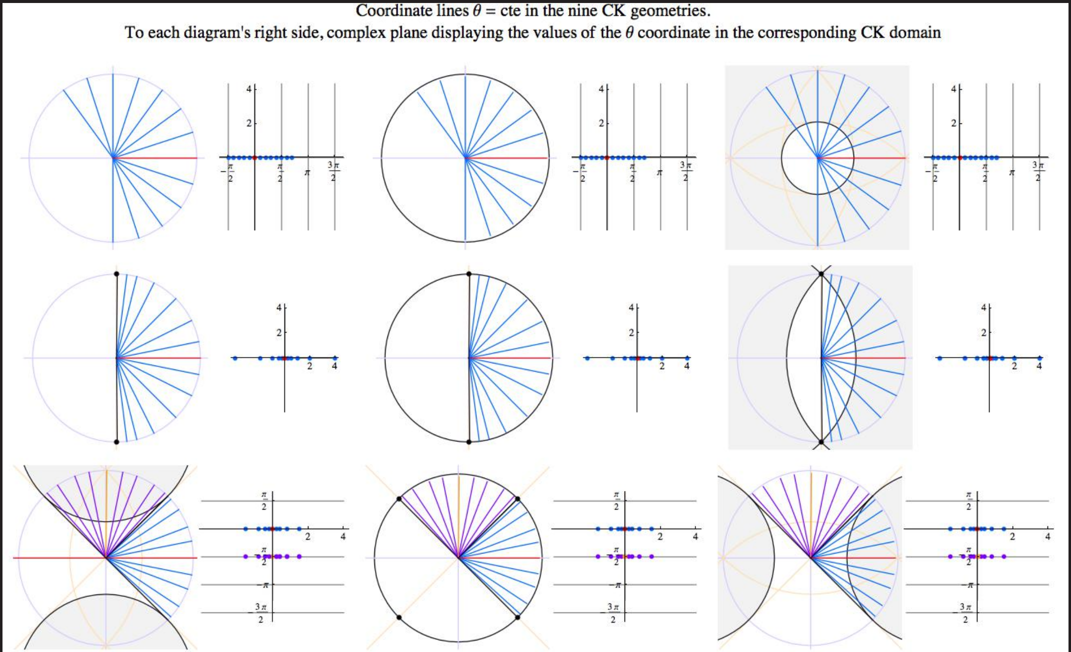




Coordinate lines r = cte in the nine CK spaces



✓ The coordinate lines r=cte are circles with center at O_0 in the geometry of each CK space

Coordinate lines θ = cte in the nine CK spaces



✓ The coordinate lines θ=cte are geodesics through O_0 in the geometry of each CK space

The duality in the CK scheme [1]

- **Duality is an interchange between the basic elements in the CK original space and the ones in the dual, according to:**

Dual CK space $\mathcal{D}(\mathbf{S})$	*versus*	Original CK space $\mathbf{S}$
Points (invariant under $\mathcal{J}=-P_1$)	•	Actual lines (invariant under P_1)
Distance between points (along $\mathcal{P}_1=-J$)	•	Angle between actual lines (along J)
Actual lines (invariant under $\mathcal{P}_1=-J$)	•	Points (invariant under J)
Angle between actual lines (along $\mathcal{J}=-P_1$)	•	Distance between points (along P_1)
Ideal lines (invariant under $\mathcal{P}_2=-P_2$)	•	Ideal lines (invariant under P_2)
Angle between ideal lines (along $\mathcal{J}=-P_1$)	•	Actual distance between ideal lines (along P_1)

⋆ **The map $\mathcal{D}$** leaves the Lie algebra invariant, interchanges the two constants $\kappa_1 \leftrightarrow \kappa_2$, and hence the space of points with the space of (actual) lines, $S^2_{[\kappa_1],\kappa_2} \leftrightarrow S^2_{\kappa_1,[\kappa_2]}$.

⋆ **In the sphere S^2 this is the well known polarity.**

⋆ **Duality relates two CK geometries which are different in general.** Only when $\kappa_1=\kappa_2$ the CK geometry is self-dual. Examples: $\mathbf{S}^2, \mathbf{G}^{1+1}, \mathbf{dS}^{1+1}$.

- **Theorem** *The dual of a CK space with curvature κ_1 and metric of signature type $(+,\kappa_2)$ is the CK space with curvature κ_2 and metric of signature type $(+,\kappa_1)$.*

The duality in the CK scheme [3]

Spherical: $\mathbf{S}^2$ $S^2_{[+],+} = SO(3)/SO(2)$	Euclidean: $\mathbf{E}^2$ $S^2_{[0],+} = ISO(2)/SO(2)$	Hyperbolic: $\mathbf{H}^2$ $S^2_{[-],+} = SO(2,1)/SO(2)$
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Anti-de Sitter: $\mathbf{AdS}^{1+1}$ (Co-Hyperbolic) $S^2_{[+],-} = SO(2,1)/SO(1,1)$	Minkowskian: $\mathbf{M}^{1+1}$ $S^2_{[0],-} = ISO(1,1)/SO(1,1)$	De Sitter: $\mathbf{dS}^{1+1}$ (Doubly Hyperbolic) $S^2_{[-],-} = SO(2,1)/SO(1,1)$

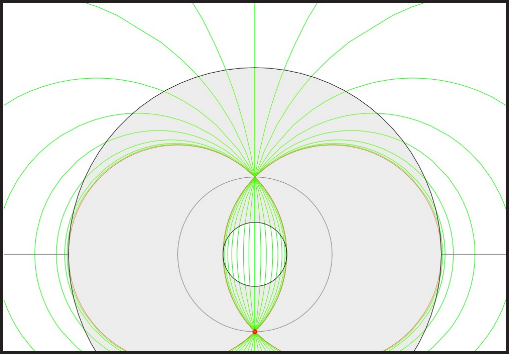
- **Duality is realized by a symmetry along the main diagonal.**

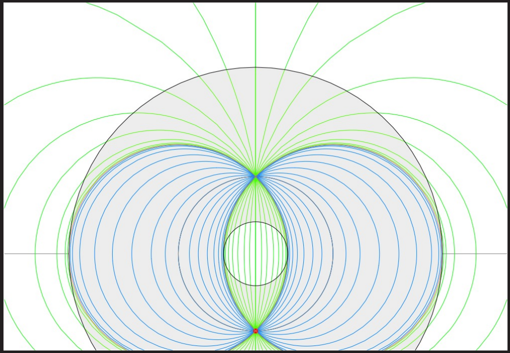
$$\mathcal{D}: \underset{(1,1)}{\mathbf{S}^2} \longleftrightarrow \underset{(1,1)}{\mathbf{S}^2}, \qquad \mathcal{D}: \underset{(-1,1)}{\mathbf{H}^2} \longleftrightarrow \underset{(1,-1)}{\mathbf{AdS}^{1+1}}, \qquad \mathcal{D}: \underset{(-1,-1)}{\mathbf{dS}^{1+1}} \longleftrightarrow \underset{(-1,-1)}{\mathbf{dS}^{1+1}}.$$

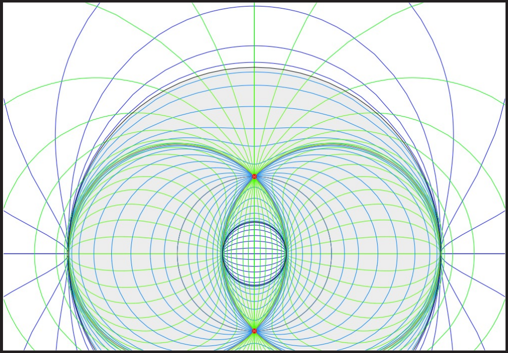
  - ⋆ **The sphere S^2 and the DeSitter space dS^{1+1} are autodual**
  - ⋆ **Hyperbolic plane H^2 and the AntiDeSitter space AdS^{1+1} are mutually dual.**

Visualizing duality in the stereocentral model: The dual of H^2 is AdS^{1+1}

- **All the lines orthogonal to l_1 meet in a single point.**
- **This point** is in the Ideal sector of $\mathbf{H}^2$, which is AdS^{1+1}.
- **Lines orthogonal to l_1** and the complete system of associated orthogonal coordinate net, covering the Actual and Ideal Sectors of $\mathbf{H}^2$.







Some applications: The classification of the confocal coordinate systems

- **Generic systems**
  - ✓ Elliptic (actual foci, actual focal separation)
  - ✓ Parabolic (one foci actual, other focus coactual, coactual focal separation)
  - ✓ CoElliptic (coactual foci, actual focal separation)
- **Limiting systems (non generic)**
- **Particular systems, for special values of the focal separation, e.g.**
  - ✓ Equiparabolic systems, with focal separation equal to a quadrant
  - ✓ Isoelliptic (two focus with isotropic separation)
  - ✓ ...
- ⋆ **Horosystems**
  - ✓ HoroElliptic (one actual focus, other focus at infinity)
  - ✓ HoroCoElliptic (one coactual focus, other focus at infinity)
- ⋆ **Coalescing foci**
  - ✓ Polar, parallel, horocyclic

A sample

• J. A. de Azcárraga, F. J. Herranz, J. C. Pérez Bueno and M. Santander: *"Central Extensions of the quasi-orthogonal Lie Algebras"*. J. Phys. A., **31**, 1373-1394, (1998). • F. J. Herranz, J. C. Pérez Bueno and M. Santander: *"Central Extensions of the families of quasi-unitary Lie Algebras"*. J. Phys. A., **31**, 5327-5347, (1998). • F. J. Herranz, and M. Santander: *"The family of quaternionic quasi-unitary Lie Algebras and their central extensions"*. J. Phys. A., **32**, 4495-4507, (1999).	Complete description of the central extensions of all Lie algebras $so_{\kappa_1\ldots\kappa_N}(N+1)$, $su_{\kappa_1\ldots\kappa_N}(N+1)$ and $sp_{\kappa_1\ldots\kappa_N}(N+1)$ in the three 'real, complex and quaternionic type' Cayley Klein series, for any N and general κ_i

• F. J. Herranz, R. Ortega, M. Santander: *"Trigonometry of space-times: a new self-dual approach to a curvature/signature (in)dependent trigonometry"*. math-ph/9910041, J. Phys. A **33** 4525-4553 (2000). • R. Ortega, M. Santander: *"Trigonometry of 'complex Hermitian' type homogeneous symmetric spaces"*. J. Phys. A., **35**, 7877-7917 (2002)	An exhaustive and complete study of trigonometry in all rank-one spaces of real and 'complex' type, both made in a completely general CK fashion, with κ_1, κ_2 and η as parameters. Real spaces are related to space-time (they include all homogeneous models of non-relativistic and relativistic space-times), and complex 'hermitian' ones include the quantum space of states.

• A. Ballesteros, F. J. Herranz, M. A. del Olmo y M. Santander: *"Quantum Structure of the Motion Groups of the Two Dimensional Cayley–Klein Geometries"*. J. Phys. A, **26**, 5801–5823, (1993). • A. Ballesteros, F. J. Herranz, M. A. del Olmo y M. Santander: *"Quantum (2+1) Kinematical Algebras: A Global Approach"*. J. Phys. A, **27**, 1283–1297, (1994).	Several papers where the CK κ_1, κ_2 scheme is used in relation to quantum deformations of the classical CK groups and algebras

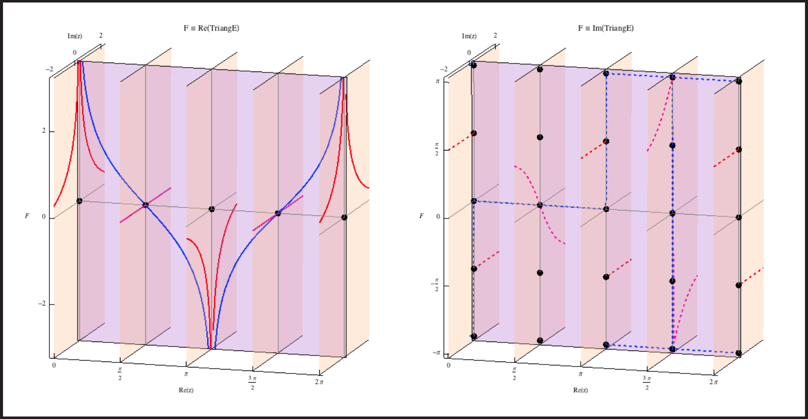
Beyond

- **The scheme encompasses all symetric homogeneous spaces**
  - ✓ Real, complex, quaternionic type symetric homogeneous spaces in any dimension, and exceptional ones as well (connection with octonions).
- ⋆ **For instance, in the complex hermitian case** (the CK general version of $su(N)$), it turns out that there are n commuting involutions). The extra involution appear as a Cayley-Dickson parameter, leading to spaces over the three composition algebras of complex numbers, Study numbers and split complex numbers.
- ⋆ **Dynamics, Integrability and superintegrability in CK spaces**

Congratulations, Miguel Angel

General Properties or propositions should be more easily demonstrable than any special case of it

- **J. I. Sylvester** Note on Spherical Harmonics, *Phil. Mag. (1876)*



Thank you very much,

Any comment, criticism, reference, . . . , welcome at **msn@fta.uva.es**